

Ch4. Limits and Continuity

§ 4.1. Limit of a function

Def (Limit) Let (X, d) be a metric space, $E \subset X$ and $f: E \rightarrow \mathbb{R}(\mathbb{C})$

Suppose p is a limit point of E , we write $\lim_{x \rightarrow p} f(x) = L$

if there is an $L \in \mathbb{R}$ satisfying the following:

for every $\varepsilon > 0$, $\exists \delta > 0$ s.t. $|f(x) - L| < \varepsilon$ if $0 < d(x, p) < \delta$ and $x \in E$.
($d(x, p) < \delta \Rightarrow |f(x) - L| < \varepsilon$)

Remark 1) $\lim_{x \rightarrow p} f(x)$ is independent of $f(p)$. (not required that $p \in E$)

2) $\lim_{x \rightarrow p} f(x)$ is unique (if exists)

3) Suppose $\lim_{x \rightarrow p} f(x) = L$ and $\lim_{x \rightarrow p} f(x) = L'$.

Given $\varepsilon > 0$. $\exists \delta$ s.t. $|f(x) - L| < \varepsilon/2$ if $0 < d(x, p) < \delta$. $x \in E$.

$\exists \delta'$ s.t. $|f(x) - L'| < \varepsilon/2$ if $0 < d(x, p) < \delta'$. $x \in E$.

Let $x \in E$ such that $0 < d(x, p) < \min(\delta, \delta')$

$$|L - L'| \leq |f(x) - L'| + |f(x) - L| < \varepsilon \Rightarrow |L - L'| < \varepsilon \Rightarrow L = L'$$

3) $f: [0, 1] \rightarrow \mathbb{R}$, $f(x) = \begin{cases} 0 & \text{if } x \in \mathbb{Q} \\ x & \text{if } x \notin \mathbb{Q} \end{cases}$

$$\rightarrow \lim_{x \rightarrow 0} f(x) = 0.$$

if $x \neq 0$, $\lim_{x \rightarrow x} f(x)$ is arbitrary.

4) Limit is meaningless if not limit point. (= isolated point).
(δ interval not empty...)

Thm Let $E \subset X$. $f: E \rightarrow \mathbb{R}$, p is a limit point of E .

$\lim_{x \rightarrow p} f(x) = L$ iff $\lim_{n \rightarrow \infty} f(p_n) = L$ for every seq. $\{p_n\} \subset E$, $p_n \neq p$ such that $p_n \rightarrow p$ as $n \rightarrow \infty$.

pf) ($\Rightarrow$) Let $\{p_n\} \subset E$ a sequence s.t. $p_n \neq p$ & $p_n \rightarrow p$.

Given $\varepsilon > 0$. $\exists \delta > 0$ s.t. $|f(x) - L| < \varepsilon$ if $0 < d(x, p) < \delta$. $x \in E$.

Let $N \in \mathbb{N}$ s.t. $0 < d(p_n, p) < \delta$ if $n \geq N$.

$\therefore |f(p_n) - L| < \varepsilon$ if $n \geq N$.

($\Leftarrow$) (Contrapositive) Suppose $\lim_{x \rightarrow p} f(x) \neq L$. then for some $\varepsilon > 0$,

there is no $\delta > 0$ s.t. $|f(x) - L| < \varepsilon$ even if $0 < d(x, p) < \delta$. $x \in E$.

($\Leftrightarrow$ every $\delta > 0$, $\exists x$ s.t. $|f(x) - L| \geq \varepsilon$ and $0 < d(x, p) < \delta$, $x \in E$).

In particular, taking $\delta = 1/n$, $\exists p_n \in E$ s.t. $|f(p_n) - L| \geq \varepsilon$, and $0 < d(p_n, p) < 1/n$, $p_n \in E$.

$\lim_{n \rightarrow \infty} p_n = p$, $p_n \in E$, but $\lim_{n \rightarrow \infty} f(p_n) \neq L$. $\nRightarrow$

Thm Let $E \subset X$. $f, g: E \rightarrow \mathbb{R}$ & p is a limit point of E . $\lim_{x \rightarrow p} f(x) = A$. $\lim_{x \rightarrow p} g(x) = B$.

1) $\lim_{x \rightarrow p} (f(x) + g(x)) = A + B$.

2) $\lim_{x \rightarrow p} f(x)g(x) = AB$.

3) if $B \neq 0$, $\lim_{x \rightarrow p} \frac{1}{g(x)} = \frac{1}{B}$.

$|B| \leq |B - g(x)| + |g(x)| < \frac{|B|}{2} + |g(x)|$

pf) $\exists \delta > 0$ s.t. $|g(x) - B| < \frac{|B|}{2}$ if $d(x, p) < \delta$. $x \in E$.

given $\varepsilon > 0$. $\exists \delta' > 0$ s.t. $|g(x) - B| < \frac{|B|^2}{2} \varepsilon$ if $d(x, p) < \delta'$. $x \in E$. if $d(x, p) < \delta$. $x \in E$.

$\left| \frac{1}{g(x)} - \frac{1}{B} \right| = \frac{|B - g(x)|}{|g(x)||B|} \leq \frac{2}{|B|^2} |B - g(x)| < \varepsilon$ if $d(x, p) < \delta'$. $x \in E$.

Def (Limit at ∞) Let $E \subset \mathbb{R}$. $f: E \rightarrow \mathbb{R}$ such that

$E \cap (a, \infty) \neq \emptyset$ for all $a > 0$. Then
($(-\infty, a) \neq \emptyset$)

we write $\lim_{\substack{x \rightarrow \infty \\ (x \rightarrow -\infty)}} f(x) = L$ if there is an $L \in \mathbb{R}$ satisfying
: every $\varepsilon > 0$, $\exists M > 0$ s.t. $|f(x) - L| < \varepsilon$ if $x \geq M$, $x \in E$.
($x \leq -M$)

Ex $f(x) = \frac{1}{x}$. $E = \mathbb{R} \setminus \{0\}$. $\rightarrow \lim_{x \rightarrow \infty} f(x) = 0$.

$\rightarrow$ Let $\varepsilon > 0$ be given. Choose $M > 0$ such that $\frac{1}{M} < \varepsilon$.

$$|f(x) - 0| = \frac{1}{|x|} < \varepsilon \text{ if } x \geq M.$$

§ 4.2. Continuous Functions.

Def Let $E \subset X$. $f: E \rightarrow \mathbb{R}(\infty)$ and $p \in E$. We say

f is continuous at p if for every $\epsilon > 0$, there is $\delta > 0$ st.

$$|f(x) - f(p)| < \varepsilon \quad \text{if} \quad d(x, p) < \delta, \quad x \in E.$$

Remark: 1) If p is a limit point of E , $f \in \mathbb{R}$ continuous at p iff $\lim_{x \rightarrow p} f(x) = f(p)$

2) If p is an isolated point of E , any f is continuous at p .

$$\underline{\text{Ex}}_3) f: [0, 1] \rightarrow \mathbb{R}, \quad f(x) = \begin{cases} 0 & x \in \mathbb{Q} \\ x & x \notin \mathbb{Q} \end{cases}$$

$\rightarrow f$ is continuous at 0 ($\because f(0) = \lim_{x \rightarrow 0} f(x) = 0$)

f is discontinuous at $x \notin \mathbb{Q}$ ($\because$ ~~극한~~ ~~한라지~~ $\frac{0}{0}$).

4) $f: [0, 1] \rightarrow \mathbb{R}$ $f = \begin{cases} x & \text{if } x \notin \mathbb{Q} \\ \frac{1}{n} & \text{if } x = \frac{m}{n} \text{ (in lowest terms)} \end{cases}$

$\rightarrow f$ is discontinuous at x if $x \in \mathbb{Q}$ (한 칸 한 칸 $\therefore x = \frac{m}{n} + \frac{\sqrt{2}}{k} \epsilon$)

but f is continuous at x if $x \notin \mathbb{Q}$.

pf) Let $p \in \mathbb{Q}$ & $p \in [0, 1]$. Given $\varepsilon > 0$. Choose N st. $\frac{1}{N} < \varepsilon$.

$$S = \left\{ \frac{m}{a} \in [0, 1] : m, n \text{ has no common factor} \right\}$$

$$\delta = \min_{s \in S} |p-s| \neq 0 \quad (\because p \notin \mathbb{Q})$$

if $|x - p| < \delta$, $|f(x) - f(p)| = |f(x)| < \frac{1}{\sqrt{x}} < \varepsilon$

↪ $\begin{cases} \text{if } x \text{ is irrational} \rightarrow 0 \\ x \text{ is rational} \rightarrow \frac{1}{x^2} \text{ 이 됨.} \end{cases}$

* $\mathbb{R}$ is a metric space

Topological characterization of Continuity.

more easy open sets easy!

thm Let $E \subset X$. & $f: E \rightarrow \mathbb{R}$ then f is continuous on E iff
 $f^{-1}(V)$ is open in E for every open set $V \subset \mathbb{R}$.
Eel se point nahi aye.

pt) $(\Rightarrow)$ Let $V \subset \mathbb{R}$ be open & let $p \in f^{-1}(V)$. need to show $N_\epsilon(p) \subset f^{-1}(V)$.

$f(p) \in V$. Since V is open, $\exists \epsilon > 0$ s.t. $N_\epsilon(f(p)) \subset V$.

Since f is continuous on E . $\exists \delta > 0$ s.t.

$$|f(x) - f(p)| < \epsilon \text{ if } d(x, p) < \delta \text{ \& } x \in E. \quad \text{--- (*)}$$

$$(\Leftarrow f(N_\delta(p) \cap E) \subset N_\epsilon(f(p))).$$

$$\Leftrightarrow f(x) \in N_\epsilon(f(p)) \text{ for all } x \in N_\delta(p) \cap E.$$

$$\star \hookrightarrow \underbrace{N_\delta(p) \cap E \subset f^{-1}(N_\epsilon(f(p)))}_{\text{need to show}}$$

$(\Leftarrow)$ $\star \frac{1}{2}$ $\frac{1}{2}$ stat.

Let $p \in E$. Let $\epsilon > 0$ to be given. (need to show continuity on p).

By assumption, $f^{-1}(N_\epsilon(f(p)))$ is open. (in E) $p \in f^{-1}(N_\epsilon(f(p)))$.

Since $f^{-1}(N_\epsilon(f(p)))$ is open, (in E) $\exists \delta > 0$ s.t. $N_\delta(p) \cap E \subset f^{-1}(N_\epsilon(f(p)))$.

$$\Rightarrow f(N_\delta(p) \cap E) \subset N_\epsilon(f(p)) \rightarrow (*)$$

Cor f is continuous on E iff $f^{-1}(W)$ is closed in E
 for every closed set $W \subset \mathbb{R}$

pt) Let V be open

$$\underbrace{f^{-1}(V^c)}_{\text{closed}} = E \setminus \underbrace{f^{-1}(V)}_{\text{open}}$$

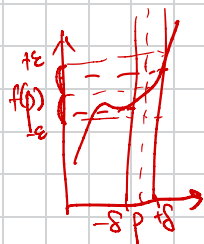
$$= \underbrace{\left(\overline{f^{-1}(V)} \right)^c}_{\text{open in } E}$$

$$\star A \subset B$$

$$\Leftrightarrow f(A) \subset f(B)$$

$$\Leftrightarrow f^{-1}(f(A)) \subset f^{-1}(f(B))$$

$$\star f^{-1}(f(A)) \supset A, \quad f(f^{-1}(B)) \subset B$$



Thm Let $K \subset X$ be compact. if $f: K \rightarrow \mathbb{R}$ is continuous, $f(K)$ is opt.
opt set of a continuous f.

pf) Let $\{V_\alpha\}_{\alpha \in A}$ be an open cover of $f(K)$.

By thm, $f^{-1}(V_\alpha)$ is open in K .

($\exists$ open V_α st. $f^{-1}(V_\alpha) = V_\alpha \cap K$).
(in X) 전체 ket every point in K $\rightarrow$ is image there

$\{V_\alpha\}$ is an open cover of K . (opt).

(Let $p \in K$. $f(p) \in f(K)$. $f(p) \in V_\alpha$ for some α).
 $\Rightarrow p \in f^{-1}(V_\alpha)$ * $= \{x \in K: f(x) \in V_\alpha\}$

Since K is opt. $\exists$ a finite subcover $V_1, \dots, V_n$

$$\text{s.t. } \bigcup_{i=1}^n V_{\alpha_i} \supset K$$

$$K = \bigcup_{i=1}^n (V_{\alpha_i} \cap K) = \bigcup_{i=1}^n f^{-1}(V_{\alpha_i})$$

$$f(K) = f\left(\bigcup_{i=1}^n f^{-1}(V_{\alpha_i})\right) \subset \bigcup_{i=1}^n V_{\alpha_i} \quad f(K) \text{ is opt.}$$

$$\bigcup_{i=1}^n f(f^{-1}(V_{\alpha_i})) \subset \bigcup_{i=1}^n V_{\alpha_i}$$

finite f of V_{α_i}

* $f(f^{-1}(A)) \subset A$.
(증거는 집합론 논리법).
 $f \circ f^{-1}$

Cor. Let $K \subset X$ be opt. & $f: K \rightarrow \mathbb{R}$ be continuous.

증-값 정리. Then $\exists p, q \in K$ such that

$$f(p) \leq f(x) \leq f(q), \quad \forall x \in K$$

pf) By thm, $f(K)$ is opt. $f(K)$ is closed & bounded ($= f(K) \subset \mathbb{R}$) 최대-최소

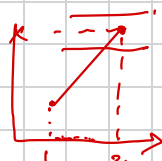
$$M = \sup_{x \in \mathbb{R}} \{f(x): x \in K\}, \quad m = \inf_{x \in \mathbb{R}} \{f(x): x \in K\}$$

$$(\text{closed of } \mathbb{R}) \quad m, M \in f(K)$$

$f(K)$ 의 limit point.

$$\exists p, q \text{ st. } m = f(p) \text{ \& } M = f(q)$$

OK



$f: K \rightarrow \mathbb{R}$ bounded $\Rightarrow \exists x_j \in K$ s.t. $\lim_{j \rightarrow \infty} f(x_j) = M$ ($\because \sup$).
 $\text{seq} \rightarrow \text{conv.}$ subseq to $p \in K$ opt. $x_j \rightarrow p \Rightarrow f(x_j) \rightarrow f(p) = M$.
 continuous sequence? $\Rightarrow f(x_j) = f(p) = M$.
Thm Let $E \subset \mathbb{R}$ be a connected set. If $f: E \rightarrow \mathbb{R}$ is continuous, then $f(E)$ is connected. $\therefore$ from seq. to max? min?

A is disconnected if there are disjoint open sets U, V s.t.
 $1) A \cap U \neq \emptyset, A \cap V \neq \emptyset, 2) A \subset U \cup V$ ($\Leftrightarrow (A \cap U) \cup (A \cap V) = A$).

$f: E \rightarrow \mathbb{R}$ Suppose $f(E)$ is disconnected.

$\exists$ disjoint open sets U, V s.t. $f(E) \cap U \neq \emptyset, f(E) \cap V \neq \emptyset$.
 $f(E) \subset U \cup V$

Since f is continuous & U, V are open, $f^{-1}(U), f^{-1}(V)$ are open.

$q) \begin{cases} \emptyset \neq f^{-1}(U) \cap E \\ \emptyset \neq f^{-1}(V) \cap E \end{cases}$
 $E \subset f^{-1}(U \cup V) = f^{-1}(U) \cup f^{-1}(V)$
 $\neq f^{-1}(A \cap B) = f^{-1}(A) \cap f^{-1}(B)$

$\textcircled{2} E \subset f^{-1}(U) \cup f^{-1}(V), f^{-1}(U) \cap f^{-1}(V) = \emptyset$

$\because U, V$ are disjoint

$\Rightarrow E$ is disconnected. $\times$

$\langle \text{proof} \rangle$

$\downarrow$ observation. Let $E \subset \mathbb{R}$ be a connected set.

If $x, y \in E$ & $x < y$, $z \in E$ for $\forall z \in (x, y)$.

$\rightarrow$ Suppose $\exists z \in (x, y)$ s.t. $z \notin E$.



$\Rightarrow E$ is disconnected. $\times$

$V_1 \cap V_2 = \emptyset$

$f^{-1}(V_1 \cap V_2) = \emptyset$
 $= f^{-1}(V_1) \cap f^{-1}(V_2)$

(I.V.T)

Cor. Let $f: [a, b] \rightarrow \mathbb{R}$ be a continuous fun.

suppose $f(a) < r < f(b)$. then $\exists c \in (a, b)$ s.t. $f(c) = r$.

pf) By thm. $f([a, b])$ is connected. (f is conti.)

Since $f(a), f(b) \in f([a, b])$ & $f(a) < r < f(b)$.

$r \in f([a, b])$. thus $\exists c \in [a, b]$ s.t. $f(c) = r$.

$\underbrace{a < b}_{0 \leq 2}$

($f(a) \neq r$,

$f(b) \neq r$)

Let n be a positive integer.

Ex1. For $\forall \epsilon > 0$. $\exists \delta > 0$ s.t. $y^n = r$.

$f(x) = x^n - r$. f continuous. $\star$ $\frac{1}{n}$ is not integer.

$f([0, 1+r]) \rightarrow \mathbb{R}$ $f(0) = -r < 0 < f(1+r) = \underline{(1+r)^n - r} > 0$.

$\therefore \exists y \in (0, 1+r)$ s.t. $f(y) = 0$.

Ex2 Let $f: [0, 1] \rightarrow [0, 1]$ be continuous.

then $\exists x \in [0, 1]$ s.t. $f(x) = x$. $\Rightarrow$ '고정점'.

$\Rightarrow$ We may assume $f(0) \neq 0$ and $f(1) \neq 1$.

consider $g(x) = f(x) - x$. ~~$g: [0, 1] \rightarrow [0, 1]$~~

$g(0) = f(0) > 0 > g(1) = f(1) - 1$

$\exists c \in (0, 1)$ s.t. $g(c) = 0$.

§ 4.3. Uniform Continuity $\Rightarrow$ 연속성보다 각 점에 대한 연속성.

Def Let $E \subset X$ and $f: E \rightarrow \mathbb{R}$. We say f is uniformly continuous on E if for every $\varepsilon > 0$, there is a $\delta > 0$ s.t.

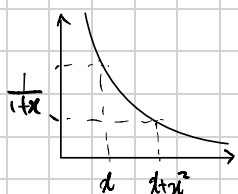
$$|f(x) - f(y)| < \varepsilon \text{ if } d(x, y) < \delta \text{ \& } x, y \in E.$$

(remark. $|f(x) - f(p)| < \varepsilon$ if $d(x, p) < \delta(p)$, $x \in E$).

Ex 1) $f: \mathbb{R} \rightarrow \mathbb{R}$
 $f(x) = x$. $|f(x) - f(y)| = |x - y| < \varepsilon$ if $|x - y| < \left(\frac{\varepsilon}{2}\right) = \delta$.

2) $f: (0, 1) \rightarrow \mathbb{R}$
 $f(x) = \frac{1}{x}$. $f(x) - f(x^2) = \frac{1}{x} - \frac{1}{x^2} = \frac{x}{x^2} = \frac{1}{x} > \frac{1}{2}$.

$\rightarrow \varepsilon = \frac{1}{2}$ 일 때 $|x^2 - x|$ 도 $\frac{1}{2}$ 이하가 안 됨.



$\cdot \exists \delta > 0$ s.t. $|f(x) - f(y)| < \frac{1}{2}$ if $|x - y| < \delta$, $x, y \in (0, 1)$

$\rightarrow x = \frac{\sqrt{\delta}}{2}, y = \frac{\delta}{4} + \frac{\sqrt{\delta}}{2} \geq \frac{1}{2}$ 일 때 $(|f(x) - f(y)| > \frac{1}{2})$

3) $|f(x) - f(y)| \leq c|x - y|$, $\forall x, y \in E$.

($\Leftrightarrow |f(x) - f(y)| < \varepsilon$ if $d(x, y) < \frac{\varepsilon}{2c}$)

$\rightarrow$ We say f is Lipschitz-continuous.

($\times f(x) = \frac{1}{x}$, $f: [10^{-2}, 1] \rightarrow \mathbb{R}$ is not Lipschitz)

Thm Let $K \subset X$ be compact. If $f: K \rightarrow \mathbb{R}$ is continuous, then f is uniformly continuous.

pf) Since f is continuous, given $\varepsilon > 0$, for every $p \in K$

$\exists \delta(p) > 0$ s.t. $|f(x) - f(p)| < \frac{\varepsilon}{2}$ if $d(x, p) < \delta(p)$, $x \in K$.

$\bigcup_{p \in K} N_{\delta(p)/2}(p) \supset K$. Since K is cpt,

$$\Rightarrow N_{\delta(p)/2}(p) \dots N_{\delta(p_n)/2}(p_n) \subseteq \bigcup_{j=1}^n N_{\delta(p_j)/2}(p_j) \supset K.$$

$$\text{Set } \delta = \min \left(\frac{\delta(p)}{2}, \dots, \frac{\delta(p_n)}{2} \right).$$

Then if $d(x, y) < \delta$, $x, y \in K$, $x, y \in N_{\delta(p_j)/2}(p_j)$ for some j .

$$|f(x) - f(y)| \leq |f(x) - f(p_j)| + |f(p_j) - f(y)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \quad \text{if}$$

$$d(x, y) < \delta, \quad x, y \in K.$$

§ 4.4. Monotone Functions and Discontinuities.

$\hookrightarrow \mathbb{R}$.

Def (right limit) Let $E \subset \mathbb{R}$ & $f: E \rightarrow \mathbb{R}$.

Suppose p is a limit point of $E \cap (p, \infty)$. Then we say f has a right limit if there is an $L \in \mathbb{R}$ with the following property.

for every $\varepsilon > 0$, $\exists \delta > 0$ s.t. $|f(x) - L| < \varepsilon$ if $\underbrace{p < x < p + \delta}_{\text{right limit}}$, $x \in E$.

(left limit) p is a limit point of $E \cap (-\infty, p)$...

$$f(p+) = \lim_{x \rightarrow p+} f(x). \quad f(p-) = \lim_{x \rightarrow p-} f(x).$$

$\hookrightarrow$ right limit

$\hookrightarrow$ left limit.

Def Let $p \in E$ & $f: E \rightarrow \mathbb{R}$. We say f is right continuous at p if

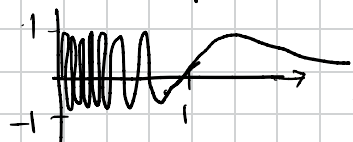
for every $\varepsilon > 0$, $\exists \delta > 0$ s.t. $|f(x) - f(p)| < \varepsilon$ if $\underbrace{p \leq x < p + \delta}_{\text{right continuity}}$, $x \in E$.

$$\Leftrightarrow f(p+) = f(p)$$

left continuity at p is defined similarly. $\Leftrightarrow f(p-) = f(p)$.

Remark. f is continuous at $p \Leftrightarrow f(p+) = f(p) = f(p-)$.

Ex 1) $f(x) = \begin{cases} 1 & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases} \Rightarrow f(0+) \neq f(0-).$
 $f(0+) = f(p)?$ f is right cont. at $p=0$.

2) $f(x) = \begin{cases} \sin \frac{\pi}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$

 $\rightarrow f(0+) \text{ not exist.}$

Monotone fns.

Def Let I be an interval & $f: I \rightarrow \mathbb{R}$. We say f is
monotone increasing (decreasing) if

$$f(x) \leq f(y) \text{ for } x < y, \quad x, y \in I.$$

$$(f(x) \geq f(y))$$

$\rightarrow f$ is monotonic decreasing...
 $\text{increasing/decreasing}$

Thm Let $I \subset \mathbb{R}$ be an open interval & $f: I \rightarrow \mathbb{R}$ be a monotonic
 increasing function. Then $f(p+)$, $f(p-)$ exist for all $p \in I$.

$$\sup_{x < p} f(x) = f(p-) \leq f(p) \leq f(p+) = \inf_{x > p} f(x).$$

pf) $\{f(x) : x > p, x \in I\}$ bounded below ($f(p)$)

$$\exists \inf_{x > p} f(x) = A. \quad \text{claim: } A = f(p+). \quad (*f(p) \in A)$$

Given $\varepsilon > 0$. $\exists \delta_0 > p$ s.t. $f(x_0) < A + \varepsilon$. $x_0 \in I$.

$$A \leq f(x) \leq f(x_0) < A + \varepsilon \text{ if } p < x < \textcircled{\delta_0} \rightarrow x_0 = p + \delta.$$

($\because$ monotonic)

Take $\delta = \delta_0 - p > 0$, then $|f(x) - A| < \varepsilon$ if $p < x < p + \delta$.

$\Rightarrow$ monotonically increasing fn: $\lim_{x \rightarrow p+} f(x)$ exists!
 $\rightarrow \lim_{x \rightarrow p-} f(x)$ also exists.

$\rightarrow$ monotonic fns are not necessarily continuous.

Def Let $f: I \rightarrow \mathbb{R}$, & $p \in I$.

We say f has a simple discontinuity at p if f is discontinuous at p

& $f(p+)$ & $f(p-)$ exists if $p \in \text{int } I$.

$f(p-)$ exists if p is the right endpoint ($f(p) \neq f(p-)$)
 $f(p+)$ exists if p is the left endpoint ($f(p) \neq f(p+)$)

Cor Let I be an open interval & $f: I \rightarrow \mathbb{R}$ be monotone increasing.

Then every discontinuity of f is simple. Moreover, the set of points at which f is discontinuous is at most countable.

pf) Let $E = \{p \in I : f \text{ is discontinuous at } p\}$.

Let $p \in E$. Since f is discontinuous at p , $f(p-) < f(p+)$.

($\because f(p-) \leq f(p) \leq f(p+) \rightarrow$ ~~ฟังก์ชันต่อเนื่อง~~ continuous).

For each $p \in E$, choose $r_p \in \mathbb{Q}$ st. $f(p-) < r_p < f(p+)$

$p \mapsto r_p$ is $1-1$, E is at most countable.

($E \rightarrow \mathbb{Q}$)

$\rightarrow p, q \in E$. if $p < q$, $f(p+) \leq f(q-)$

($\because$ let $p < r < q \rightarrow f(p+) \leq f(r) \leq f(q-)$).

$\hookrightarrow$ ~~ฟังก์ชันต่อเนื่อง~~ ...

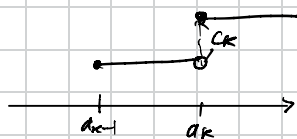
$$\begin{aligned} f(p+) &= \inf\{f(x) : x > p, x \in I\} \leq \inf\{f(x) : p < x < q\} \\ &\leq \sup\{f(x) : p < x < q\} \leq \sup\{f(x) : x < q, x \in I\} = f(q-). \end{aligned}$$

Discontinuity on a dense subset.

$$I(x) = \begin{cases} 1 & x \geq 0 \\ 0 & x < 0 \end{cases} \rightarrow \text{simple discontinuity at } x=0. \\ \text{\& right continuous}$$

$$\{d_k\}_{k=1}^n \subset \mathbb{R} : d_k \text{ are distinct.}$$

$$\{C_k\}_{k=1}^{\infty} \subset (0, \infty) : C_k > 0.$$



$$f(x) = \sum_{k=1}^{\infty} C_k \cdot I(x - d_k) \rightarrow \begin{cases} \text{continuous on } \mathbb{R} \setminus \{d_1, \dots, d_n\} \\ \text{discontinuous at } x = d_1, \dots, d_n. \\ f(d_k+) - f(d_k-) = C_k. \end{cases}$$

Thm Let $a < b$ & $\{d_k\}_{k=1}^{\infty} \subset (a, b)$ (d_k are distinct).

Let $\{C_k\}_{k=1}^{\infty}$ be a seq. s.t. $C_k > 0$ & $\sum_{k=1}^{\infty} C_k < +\infty$.

$$\text{Set } f(x) = \sum_{k=1}^{\infty} C_k \cdot I(x - d_k).$$

$$\Rightarrow 1) f(a) = 0 \text{ \& } f(b) = \sum_{k=1}^{\infty} C_k.$$

2) f is continuous on $\mathbb{R} \setminus \{d_k\}_{k=1}^{\infty}$.

3) f is discontinuous at $x = d_1, \dots, d_k$ & $f(d_k+) - f(d_k-) = C_k$

pf. 2) Let $p \in \mathbb{R} \setminus \{d_k\}_{k=1}^{\infty}$.

Given $\varepsilon > 0$. Choose N s.t. $\sum_{k=N+1}^{\infty} C_k < \varepsilon/3$.

Set $f_N(x) = \sum_{k=1}^N C_k I(x - d_k)$, $f_N(x)$ is continuous at $x = p$.

Choose $\delta > 0$ s.t. $|f_N(x) - f_N(p)| < \varepsilon/3$ if $|x - p| < \delta$.

$$|f(x) - f(p)| = \underbrace{|f_N(x) - f_N(p)|}_{f_N(x)} + \underbrace{\left| \sum_{k=N+1}^{\infty} C_k I(x - d_k) - \sum_{k=N+1}^{\infty} C_k I(p - d_k) \right|}_{f(x)}$$

$$\leq |f_N(x) - f_N(p)| + \left| \sum_{k=N+1}^{\infty} C_k I(x - d_k) \right| + \left| \sum_{k=N+1}^{\infty} C_k I(p - d_k) \right|.$$

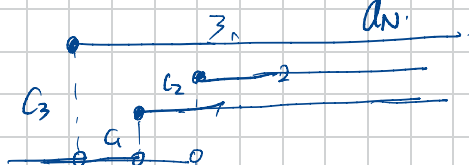
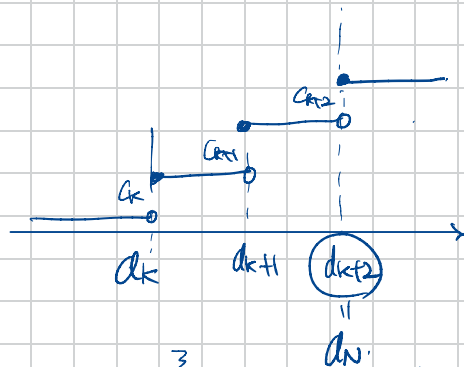
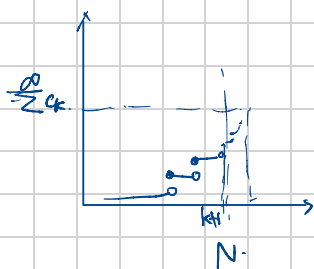
$$< \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon \quad \text{if } |x - p| < \delta.$$

3) Claim: $|f(d_{k+1}) - f(d_k) - c_k| < \varepsilon$, $\forall \varepsilon > 0$. $\rightarrow$ given $\varepsilon > 0$.

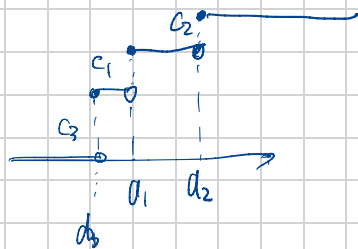
Choose N s.t. $\sum_{k=1}^{\infty} c_k < \varepsilon/3$ & $N \geq k_H$

Set $f_N(x) = \sum_{k=1}^N c_k \mathbb{I}(x - d_k)$.

$$|f(d_{k+1}) - (f(d_k) + c_k)| \leq |f_N(d_{k+1}) - f_N(d_k) - c_k| + \left| \sum_{k=1}^{\infty} c_k \mathbb{I}(x - d_k) \right|$$



$\Rightarrow$



Ch 5. Differentiation ($f: I \rightarrow \mathbb{R}$)

Def Let $f: [a, b] \rightarrow \mathbb{R}$ & $p \in [a, b]$. Then define $f'(p) = \lim_{x \rightarrow p} \frac{f(x) - f(p)}{x - p}$

provided that the limit exists. f is differentiable on $E \subset [a, b]$

if f is differentiable at every $p \in E$.

$$\text{If } p \neq b, f'(p) = \lim_{x \rightarrow p^+} \frac{f(x) - f(p)}{x - p}, \quad \text{if } p \neq a, f'(p) = \lim_{x \rightarrow p^-} \frac{f(x) - f(p)}{x - p}.$$

Remark $f'(a) = f'_+(a)$, & $f'(b) = f'_-(b)$: always agree right/left limit if exist.

Thm. Let $f: [a, b] \rightarrow \mathbb{R}$ & $p \in [a, b]$. If f is differentiable at p , then f is continuous at p .

$$\text{pf)} \quad f(x) - f(p) = \underbrace{\frac{f(x) - f(p)}{x - p}}_{f'(p)} \underbrace{(x - p)}_0 \rightarrow 0 \quad \text{as } x \rightarrow p. \quad \Leftrightarrow f(x) = f(p).$$

Thm Let $f, g: [a, b] \rightarrow \mathbb{R}$. If f, g are differentiable at p ,

$$\text{Then } 1) (f+g)'(p) = f'(p) + g'(p)$$

$$2) (fg)'(p) = f'(p)g(p) + f(p)g'(p)$$

$$3) \text{ if } g'(p) \neq 0, \left(\frac{f}{g}\right)'(p) = -\frac{f'(p)}{g(p)^2}$$

$$\begin{aligned} \left(\frac{f}{g}\right)'(p) &= \frac{f'(p)}{g(p)} - \frac{f(p)g'(p)}{(g(p))^2} \\ &= \frac{f'(p)g(p) - f(p)g'(p)}{(g(p))^2} \end{aligned}$$

$$\text{pf)} \quad \frac{\frac{f(x)}{g(x)} - \frac{f(p)}{g(p)}}{x - p} = \frac{-\frac{(g(x) - g(p))}{g(x)g(p)} \cdot (x - p)}{g(p)^2}$$

$$\lim_{x \rightarrow p} \frac{\frac{f(x)}{g(x)} - \frac{f(p)}{g(p)}}{x - p} = \frac{-g'(p)}{(g(p))^2}$$

Observation: Suppose f is differentiable at p . ($\Leftrightarrow u(x) \rightarrow 0$ as $x \rightarrow p$).

$$u(x) = \frac{f(x) - f(p)}{x - p} - f'(p) \quad (x \neq p)$$

$$u(x)(x-p) = \overbrace{0}^{\text{small } - 0 \text{ notation}} (x-p)$$

$$f(x) = f(p) + (f'(p) + u(x))(x-p) = f(p) + f'(p)(x-p) + \underbrace{\varepsilon(x)}_{\substack{f(x)-f(p) \\ x-p = f'(p) + u(x) \rightarrow 0}}$$

Thm (Chain Rule) Let $f: [a, b] \rightarrow \mathbb{R}$ & $g: \mathbb{R} \rightarrow \mathbb{R}$ with $\mathbb{I} \supset \text{Range } f$.

Suppose f is differentiable at p & g is differentiable at $q = f(p)$.

$$\text{Then } (g \circ f)'(p) = g'(f(p)) f'(p)$$

$$pf) \quad f(x) = f(p) + (f'(p) + u(x))(x-p) \quad x \neq p. \quad u(x) \rightarrow 0 \text{ as } x \rightarrow p.$$

$$g(y) = g(q) + (g'(q) + v(y))(y-q) \quad y \neq q. \quad v(y) \rightarrow 0 \text{ as } y \rightarrow q.$$

$$\underbrace{(g \circ f)(x) - (g \circ f)(p)}_{x-p} = (g'(f(p)) + \underbrace{v(f(x))}_{\downarrow 0})(\underbrace{f(p) + u(x)}_{\downarrow 0} - p)$$

$$\lim_{x \rightarrow p} \frac{(g \circ f)(x) - (g \circ f)(p)}{x-p} = g'(f(p)) f'(p).$$

§ 5.2. Mean Value Theorem.

Def Let $E \subset \mathbb{R}$, $f: E \rightarrow \mathbb{R}$. We say f has a local maximum (minimum) at $p \in E$ if $\exists \delta > 0$ s.t. $f(x) \leq f(p) \quad \forall x \in N_\delta(p) \cap E$.
 $(f(x) \geq f(p)) \quad \forall x \in N_\delta(p) \cap E$

We say f has a maximum (minimum) at p if $f(x) \leq f(p)$, $\forall x \in E$.
 $(f(p) \leq f(x))$

Thm Let $f: [a, b] \rightarrow \mathbb{R}$. Suppose f has a local maximum at $p \in (a, b)$ and differentiable at p . Then $f'(p) = 0$.

pf) Since f has a local max. at p , $\exists \delta > 0$ s.t.
 $f(x) \leq f(p)$ if $p - \delta < x < p + \delta$ & $(p - \delta, p + \delta) \subset [a, b]$.

$$\frac{f(p-h) - f(p)}{-h} \geq 0 \quad \forall 0 < h < \delta, \quad \frac{f(p+h) - f(p)}{h} \leq 0 \quad \forall 0 < h < \delta.$$

Since $f'(p)$ exists, $f'_-(p) = f'_+(p) = f'(p)$ $f'_-(p) \geq 0$ & $f'_+(p) \leq 0$

Thm (Rolle's Thm) Suppose $f: [a, b] \rightarrow \mathbb{R}$ is continuous and differentiable on (a, b) & $f(a) = f(b)$. Then $\exists c \in (a, b)$ s.t. $f'(c) = 0$.

pf) If f is a constant fn, nothing to prove.

We assume f is not a constant fn. Then $f(a) = f(b) < f(t)$ or $f(a) = f(b) > f(t)$ for some $t \in (a, b)$

If $f(a) < f(t)$ for some $t \in (a, b)$, f has a maximum. ($\because$ continuous $\rightarrow f|_{[a, b]}$ is ctd) at $c \in (a, b)$

By thm, $f'(c) = 0$.

If $f(a) > f(t)$ for some $t \in (a, b)$, f has a minimum at $c \in (a, b)$

By thm, $f'(c) = 0$.

Cor (Mean Value Theorem). Suppose $f: [a, b] \rightarrow \mathbb{R}$ is continuous on $[a, b]$ & differentiable on (a, b) . Then $\exists c \in (a, b)$ s.t. $f'(c) = \frac{f(b) - f(a)}{b - a}$.

pf) Consider $g(x) = f(x) - f(a) - \frac{f(b) - f(a)}{b - a}(x - a)$.

$0 = g(a) = g(b)$. By Rolle's thm, $\exists c \in (a, b)$ s.t. $g'(c) = 0$.

$$(\because f'(c) - \frac{f(b) - f(a)}{b - a} = 0 \therefore)$$

Cor (Cauchy Mean Value thm) Let $f, g: [a, b] \rightarrow \mathbb{R}$ be continuous & differentiable on (a, b) . Then $\exists c \in (a, b)$ s.t. $(f(b) - f(a))g'(c) = (g(b) - g(a)) \cdot f'(c)$

If $g(b) \neq g(a)$, $\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(c)}{g'(c)}$

pf) Let $h(x) = (f(b) - f(a))g(x) - (g(b) - g(a))f(x)$.

$$h(a) = f(b)g(a) - g(b)f(a) = h(b)$$

By Rolle's thm, $\exists c \in (a, b)$ s.t. $h'(c) = 0$.

Applications of M.V.T.

* If f is differentiable on (a, b) & $x_1, x_2 \in (a, b)$ & $f'(x) > 0 \quad \forall x \in (a, b)$,

$$f(x_2) - f(x_1) = f'(c)(x_2 - x_1) > 0 \rightarrow \text{increasing fun. ...}$$

Thm Let $f: [a, b] \rightarrow \mathbb{R}$. Suppose f is differentiable on $[a, b]$ & $f'(a) < \lambda < f'(b)$.

Then $\exists c \in (a, b)$ s.t. $f'(c) = \lambda$.

Cor f' has no simple discontinuity.

pf) Consider $g(t) = f(t) - \lambda t$. ($[a, b]$ on $\mathbb{R}$ at $\frac{a+b}{2}$)

$$g'(a) = f'(a) - \lambda < 0 < g'(b) = f'(b) - \lambda$$

$$g(t) = g(a) + \underbrace{(g'(a) + u(t))}_{\text{if } t \rightarrow a} (t - a). \quad \left(\lim_{t \rightarrow a} u(t) = 0 \right)$$

$$\exists t_1 > a \text{ s.t. } g(t_1) < g(a). \quad \text{Similarly } \exists t_2 < b \text{ s.t. } g(t_2) < g(b)$$

$\therefore g$ has minimum at $c \in (a, b)$. By thm, $g'(c) = 0$.

$\therefore g$ decreasing on $[a, t_1]$

g increasing on $[t_2, b]$

Thm (L'Hospital's Rule) Let $-a < a < b < \infty$. Suppose f, g are real-valued &

differentiable on (a, b) with $g'(a) \neq 0$. Suppose $\lim_{x \rightarrow a^+} \frac{f(x)}{g(x)} = L \in \mathbb{R}$.

Then if $\lim_{x \rightarrow a^+} f(x) = 0$, $\lim_{x \rightarrow a^+} g(x) = 0$, $\lim_{x \rightarrow a^+} \frac{f(x)}{g(x)} = L$. * $\nexists g(x)$ $a < x < a + \delta$ only $\exists \frac{f(x)}{g(x)}$.

Remark. $\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = L \Leftrightarrow \lim_{x \rightarrow a} f(x) = 0, \lim_{x \rightarrow a} g(x) = 0, \lim_{x \rightarrow a} \frac{f(x)}{g(x)} = L$.

pf) Let $\varepsilon > 0$. (want $\delta > 0$ s.t. $|\frac{f(x)}{g(x)} - L| < \varepsilon$ if $a < x < a + \delta$)

Since $\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = L$, $\exists \delta > 0$ s.t. $|\frac{f(x)}{g(x)} - L| < \varepsilon/2$ if $a < x < a + \delta$.

Let $a < y < x < a + \delta$, then $\exists c \in (y, x)$ s.t. $|\frac{f(x) - f(y)}{g(x) - g(y)} - L| = |\frac{f'(c)}{g'(c)} - L| < \varepsilon/2$. Cauchy's mvt.

Let $y \rightarrow a$, $|\frac{f(x)}{g(x)} - L| \leq \varepsilon/2 < \varepsilon$ if $a < x < a + \delta$.

$\Rightarrow f(y) \rightarrow 0$.

Ch 6. Integration.

§ 6.1. Riemann Integration

Def Let $a < b$. We say $P = \{x_0, x_1, \dots, x_n\}$ be a partition of $[a, b]$ if

$$a = x_0 < x_1 < x_2 < \dots < x_n = b. \quad \Delta_i := x_i - x_{i-1}, \quad i=1, \dots, n.$$

Let f be a bounded fun on $[a, b]$.

$$\left(\exists M > 0 \text{ s.t. } |f(x)| \leq M \quad \forall x \in [a, b] \right)$$

$$M_2 = \sup_{x \in [x_{i-1}, x_i]} f(x). \quad m_i = \inf_{x \in [x_{i-1}, x_i]} f(x)$$

upper sum

$$\text{For a given partition } P \text{ of } [a, b], \quad U(P, f) = \sum_{i=1}^n M_i \Delta_i$$

$$L(P, f) = \sum_{i=1}^n m_i \Delta_i$$

or $\Delta_1 \cdot \frac{M_1 + m_1}{2}$
 $\Delta_2 \cdot \frac{M_2 + m_2}{2}$
 $\dots$

$$\star \left(\inf_{x \in [a, b]} f \right) (b-a) \leq L(P, f) \leq U(P, f) \leq \left(\sup_{x \in [a, b]} f \right) (b-a)$$

Def (Upper Integral & Lower Integral)

$$\bar{J}_a^b = \int_a^b f(x) dx = \inf \left\{ U(P, f) : P \text{ is a partition of } [a, b] \right\}$$

$$\underline{J}_a^b = \int_a^b f(x) dx = \sup \left\{ L(P, f) : P \text{ is a partition of } [a, b] \right\}$$

$\rightarrow$ exist by Least Upper Bound Property

Def Let P^* be a partition of $[a, b]$. If $P \subset P^*$, then P^* is a refinement of P .

Observation. If P^* is a refinement of P , $L(P, f) \leq L(P^*, f) \leq U(P^*, f) \leq U(P, f)$

Thm Let f be a bounded fun on $[a, b]$, then $\underline{J}_a^b f \leq \bar{J}_a^b f$

pf. Let P, Q be partitions of $[a, b]$.

$$L(P, f) \leq L(P \cup Q, f) \leq U(P \cup Q, f) \leq U(Q, f)$$

$$\therefore L(P, f) \leq U(Q, f) \quad \forall P, Q.$$

$$\uparrow \quad \int_a^b f \leq U(Q, f) \quad \downarrow \quad \int_a^b f \leq \bar{J}_a^b f$$

Def Let f be a bounded fun on $[a, b]$. We say f is Riemann integrable

$$\text{if } \int_a^b f = \bar{\int}_a^b f \text{ and define } \int_a^b f = \int_a^b f.$$

$f \in R[a, b]$ if Riemann integrable on $[a, b]$

Ex1) $f(x) = \text{constant}$. $U(P, f) = c(b-a) = L(P, f)$

Ex2) $f: [0, 1] \rightarrow \mathbb{R}$. $f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \cap [0, 1] \\ 0 & \text{if } x \in \mathbb{Q}^c \cap [0, 1] \end{cases}$ $P = \{x_0, \dots, x_n\}$
 $\hookrightarrow M_2 = 1, m_2 = 0.$

$$U(P, f) = 1 \neq L(P, f) = 0.$$

Thm Let $f: [a, b] \rightarrow \mathbb{R}$ be a bounded fun.

$f \in R[a, b]$ iff for every $\varepsilon > 0$ $\exists$ a partition P of $[a, b]$ s.t.

(*) $U(P, f) - L(P, f) < \varepsilon$. ← inf. sup. interval etc...

pf) ($\Leftarrow$) Given $\varepsilon > 0$ $\exists$ a partition P s.t. (*) holds.

$$\bar{\int}_a^b f - \int_a^b f \leq U(P, f) - L(P, f) < \varepsilon$$

$$0 \leq \bar{\int}_a^b f - \int_a^b f < \varepsilon. \rightarrow \int_a^b f = \bar{\int}_a^b f. \quad f \in R[a, b].$$

($\Rightarrow$) Given $\varepsilon > 0$. $\exists$ a partition P_1 s.t. $U(P_1, f) < \bar{\int}_a^b f + \varepsilon/2$.

$\exists$ a partition P_2 s.t. $\int_a^b f - \varepsilon/2 < L(P_2, f)$.

$P = P_1 \cup P_2$. $U(P, f) - L(P, f) \leq U(P_1, f) - L(P_2, f) < \varepsilon$. ($\because \int_a^b f = \bar{\int}_a^b f$)

Thm 1) if $f \in C([a, b])$ $f \in R[a, b]$.

2) if f is monotone on $[a, b]$, then $f \in R[a, b]$.

pf) remark. $U(P, f) - L(P, f) = \sum (M_i - m_i) \Delta x_i$
 $\hookrightarrow \text{if } f(x) - f(x_i) < \varepsilon$
 $\hookrightarrow \text{if } f(x) - f(x_i) < \varepsilon$

Let $\varepsilon > 0$. Since f is uniformly continuous on $[a, b]$, $\exists \delta > 0$ s.t. $|f(s) - f(t)| < \varepsilon / (b-a)$ if $|s - t| < \delta$.

if $|s - t| < \delta$, $s, t \in [a, b]$.

Choose a partition $P = \{x_0, \dots, x_n\}$ s.t. $x_i - x_{i-1} < \delta$.

$$U(P, f) - L(P, f) = \sum_{i=1}^n (M_i - m_i) \Delta x_i < \frac{\varepsilon}{b-a} \sum_{i=1}^n \Delta x_i < \varepsilon. \quad \text{By thm, } f \text{ is Riemann integrable.}$$

pf₂) (countably many $\rightarrow$ $\sum_{k=1}^{\infty} \frac{\epsilon}{2^k}$)

We assume f is monotone increasing. Let $\epsilon > 0$. Choose a partition $P = f(a) \dots f(b)$

s.t. $\Delta_i = x_i - x_{i-1} < \frac{\epsilon}{f(b)-f(a)+1}$ $i=1 \dots n$

$$U(P, f) - L(P, f) = \sum_{i=1}^n (f(x_i) - f(x_{i-1})) \Delta_i < \frac{\epsilon}{f(b)-f(a)+1} \sum_{i=1}^n (f(x_i) - f(x_{i-1}))$$

: monotone increasing.

$$= \frac{\epsilon}{f(b)-f(a)+1} (f(b) - f(a)) < \epsilon.$$

thm* Let $f \in R[a, b]$ & Range $f \subset [c, d]$.

* $R, R \subset R$ in $C, C \subset C$ in $R, R \subset R$...

Suppose $\phi: [c, d] \rightarrow \mathbb{R}$ continuous, then $\phi \circ f \in R[a, b]$.

Cor If $f \in R[a, b]$, $f^2, |f|, f^n \in R[a, b]$.

$\rightarrow$ upper sum lower sum with points P is ok!

pf) obige. Given $\epsilon > 0$. $\exists$ a partition of $[a, b]$ s.t. $U(P, \phi \circ f) - L(P, \phi \circ f) < \epsilon$.

Let $K = \sup_{x \in [c, d]} |\phi|$. Since ϕ is uniformly continuous on $[c, d]$,

$\exists \delta > 0$ s.t. $|\phi(t) - \phi(s)| < \epsilon / (b-a)$ if $|t-s| < \delta$, $s, t \in [c, d]$.

f ist δ -glatte Abb. $\Rightarrow$ class vert. $\rightarrow$ $f(b) \dots f(a)$

② Since $f \in R[a, b]$, $\exists$ a partition P s.t. $U(P, f) - L(P, f) < \epsilon \delta / (b-a)$

$$= \sum_{i=1}^n (M_i - m_i) \Delta_i.$$

$\Rightarrow U(P, \phi \circ f) - L(P, \phi \circ f) = \sum_{i=1}^n (M_i^* - m_i^*) \Delta_i$ ($M_i^* = \sup_{x_{i-1}, x_i} \phi \circ f$, $m_i^* = \inf_{x_{i-1}, x_i} \phi \circ f$)

$$\sum_{i=1}^n (M_i^* - m_i^*) \Delta_i = \left(\sum_{i: M_i^* - m_i^* < \delta} + \sum_{i: M_i^* - m_i^* \geq \delta} \right) (M_i^* - m_i^*) \Delta_i$$

if $M_i - m_i < \delta$, $M_i^* - m_i^* = \sup_{t \in [x_{i-1}, x_i]} \phi(f(t)) - \inf_{s \in [x_{i-1}, x_i]} \phi(f(s)) < \frac{\epsilon}{2(b-a)+1}$

$$U(P, \phi \circ f) - L(P, \phi \circ f) < \frac{\epsilon}{2} + \sum_{i: M_i - m_i \geq \delta} (M_i^* - m_i^*) \Delta_i$$

$$< \frac{\epsilon}{2} + 2K \sum_{i: M_i - m_i \geq \delta} \Delta_i < \frac{\epsilon}{2} + 2K \cdot \frac{\epsilon}{4K+1} < \epsilon.$$

$\frac{\delta \epsilon}{4K+1} \rightarrow U(P, f) - L(P, f) = \sum_{i=1}^n (M_i - m_i) \Delta_i \geq \sum_{i: M_i - m_i \geq \delta} (M_i - m_i) \Delta_i \geq \delta \sum_{i: M_i - m_i \geq \delta} \Delta_i$

$$\frac{\epsilon}{4K+1} \geq \sum_{i: M_i - m_i \geq \delta} \Delta_i$$

Thm (Lebesgue's Thm) $f \in R[a, b]$ iff the set of discontinuities of f has a measure zero. ↳ not 100% yet...

Def $E \subset \mathbb{R}$ has a measure zero if for every $\varepsilon > 0$, $\exists$ a collection of open intervals $\{I_n\}_{n=1}^{\infty}$ s.t. $E \subset \bigcup_{n=1}^{\infty} I_n$ & $\sum_{n=1}^{\infty} |I_n| < \varepsilon$.

Ex. $E = \{x\} \rightarrow (x - \varepsilon/2, x + \varepsilon/2) \supset E$

$E = \{x_1, \dots, x_n\} \rightarrow (x_i - \varepsilon/2n, x_i + \varepsilon/2n) \supset E \rightarrow$ finite points.

§ 6.2. Properties of Integrable fns.

Thm Let $f, g \in R[a, b]$

a) $f+g \in R[a, b]$ & $\int_a^b (f+g) = \int_a^b f + \int_a^b g$

b) $cf \in R[a, b]$ & $\int_a^b cf = c \int_a^b f$.

c) $fg \in R[a, b]$

d) if $f \leq g$, then $\int_a^b f \leq \int_a^b g$.

pf of a) Given $\varepsilon > 0$. $\exists$ a partition P s.t. $\overset{\text{inf}}{U}(P, f) - \varepsilon/2 < \int_a^b f < \overset{\text{sup}}{L}(P, f) + \varepsilon/2$

$\textcircled{P} \text{ s.t. } U(\tilde{P}, g) - \varepsilon/2 < \int_a^b g < L(\tilde{P}, g) + \varepsilon/2$

$\hookrightarrow$ using refinement. $P \sqcup \tilde{P} \leq P \sqcup \tilde{P}$
 $\int_a^b f + \int_a^b g < L(P, f) + L(P, g) + \varepsilon < L(P, f+g) + \varepsilon$ ($\because \inf f + \inf g \leq \inf(f+g)$)

$U(P, f+g) - \varepsilon < U(P, f) + U(P, g) - \varepsilon < \int_a^b f + \int_a^b g$

$\int_a^b (f+g) - \varepsilon < \int_a^b f + \int_a^b g < \int_a^b (f+g) + \varepsilon$.

$\int_a^b f + \int_a^b g - \varepsilon < \int_a^b (f+g) \leq \int_a^b (f+g) < \int_a^b f + \int_a^b g + \varepsilon \quad \forall \varepsilon > 0$.

$\int_a^b (f+g) = \bar{\int}_a^b (f+g) = \int_a^b f + \int_a^b g \rightarrow f+g \in R[a, b]$.

c) $(f+g)^2 = f^2 + g^2 \in R[a, b]$. (using a)

$afg \in R[a, b]$. $fg \in R[a, b]$.

Thm Let $c \in (a, b)$. $f \in R[a, b]$ iff $f \in R[a, c]$ & $R[c, b]$.

Moreover $\int_a^b f = \int_a^c f + \int_c^b f$

pf Enough to show $\bar{\int}_a^b f = \bar{\int}_a^c f + \bar{\int}_c^b f$ ^(*) $\int_a^b f = \int_a^c f + \int_c^b f$ ^(**)

($\Rightarrow$) $\bar{\int}_a^b f = \int_a^b f \rightarrow \bar{\int}_a^c f = \int_a^c f$ & $\bar{\int}_c^b f = \int_c^b f$

$(\bar{\int}_a^c f - \int_a^c f) + (\bar{\int}_c^b f - \int_c^b f) = 0$ $\odot$

f of (**): Let P be a partition of $[a, b]$.

$P^* = P \cup \{c\}$, $P_1 = P^* \cap [a, c] \rightarrow$ partition of $[a, c]$.

$P_2 = P^* \cap [c, b] \rightarrow$ partition of $[c, b]$

i) $U(P_1, f) + U(P_2, f) = U(P^*, f) \leq U(P, f)$

$\bar{\int}_a^c f + \int_c^b f \leq U(P, f)$. taking inf. $\bar{\int}_a^c f + \int_c^b f \leq \bar{\int}_a^b f$

ii) Let P_1, P_2 be partitions of $[a, c], [c, b]$ respectively. $\times P = P_1 \cup P_2$

$\bar{\int}_a^b f \leq U(P, f) = U(P_1, f) + U(P_2, f) \leftarrow$ since P_1, P_2 .

$\bar{\int}_a^b f \leq U(P_1, f) + U(P_2, f)$

$\bar{\int}_a^b f \leq \bar{\int}_a^c f + \int_c^b f$

§ Riemann's Definition of Integral. ~~7월 2주~~ ^{7월 2주} sim.

Def Let $P = \{x_0, \dots, x_n\}$ be a partition of $[a, b]$.

$S(P, f) = \sum_{i=1}^n f(t_i) \Delta x_i$, $t_i \in [x_{i-1}, x_i]$. $\rightarrow$ 한 번 정해 주면 상한, 하한.

Def $\|P\| = \max_{1 \leq i \leq n} (x_i - x_{i-1})$
 $\nwarrow$ Norm

We write $\lim_{\|P\| \rightarrow 0} S(P, f) = I$ if for every $\epsilon > 0$, $\exists \delta > 0$ s.t.

$|S(P, f) - I| < \epsilon$ whenever $\|P\| < \delta$.

Thm ^{A)} If $f \in R[a, b]$. then $\lim_{\|P\| \rightarrow 0} S(P, f) = \int_a^b f$.

Conversely, ^{B)} if $\lim_{\|P\| \rightarrow 0} S(P, f) = I$, then $f \in R[a, b]$ & $\int_a^b f = I$.

Pt of B) Given $\varepsilon > 0$. $\exists \delta > 0$ s.t. $\underline{I} - \varepsilon < S(P, f) < \underline{I} + \varepsilon$ if $\|P\| < \delta$. (*)

Choose partition $P = \{x_0, \dots, x_n\}$ s.t. $\|P\| < \delta$.

$$M_i = \sup_{[x_{i-1}, x_i]} f, \quad m_i = \inf_{[x_{i-1}, x_i]} f$$

$$\exists t_i, T_i \in [x_{i-1}, x_i] \text{ s.t. } f(t_i) - \varepsilon < m_i \leq M_i < f(T_i) + \varepsilon$$

$$\sum (f(t_i) - \varepsilon) \Delta x_i < \sum m_i \Delta x_i < \sum M_i \Delta x_i < \sum (f(T_i) + \varepsilon) \Delta x_i$$

$$\widetilde{S}(P, f) - \varepsilon(b-a) < L(P, f) \leq U(P, f) < \widetilde{S}(P, f) + \varepsilon(b-a) \quad \downarrow (*)$$

$$\underline{I} - \varepsilon(1+b-a) < L(P, f) \leq U(P, f) < \underline{I} + \varepsilon(1+b-a)$$

$$\underline{I} - \varepsilon(1+b-a) < \int_a^b f \leq \int_a^b f < \underline{I} + \varepsilon(1+b-a)$$

$(\underline{I} - \varepsilon)$ $\uparrow$ $(\underline{I} + \varepsilon)$

Since ε is arbitrary, $\underline{I} = \int_a^b f = \int_a^b f$.

Pt of A) Let $\varepsilon > 0$ be given. Since $f \in R[a, b]$. $\exists$ a partition $P = \{x_0, \dots, x_n\}$ s.t.

$$\int_a^b f - \varepsilon < L(P, f) \leq U(P, f) < \int_a^b f + \varepsilon$$

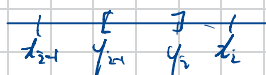
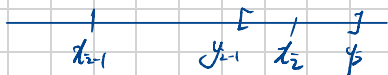
$(-)$ $\xrightarrow{2\varepsilon \text{ total}}$

$$\text{Let } \delta = \min \left(\frac{\varepsilon}{\sup_{[a,b]} (f+1)}, \min_{1 \leq i \leq n} (x_i - x_{i-1}) \right)$$

Let $\tilde{P} = \{y_0, \dots, y_l\}$ be a partition s.t. $\|\tilde{P}\| < \delta$

$$S(\tilde{P}, f) = \sum_{j=1}^l f(t_j) (y_j - y_{j-1}) \quad t_j \in [y_{j-1}, y_j], \quad j=1, \dots, l$$

$$= \sum_{\substack{j: (y_{j-1}, y_j) = x_k \\ \text{for some } k}} + \sum_{j: (y_{j-1}, y_j) \neq x_k \\ \text{for } \forall k}$$



$$\tilde{P}^* = P \cup \tilde{P} = \{ \dots, y_{i-1}, x_i, y_i, \dots \}$$

$$\text{or } \{ \dots, x_{i-1}, y_{i-1}, y_i, x_i, \dots \}$$

$$S(\tilde{P}^*, f) - \underbrace{\sum_{[a, b]} \sup f_i \cdot \delta}_{\geq L(\tilde{P}^*, f)} < S(\tilde{P}, f) < S(\tilde{P}^*, f) + \underbrace{\sum_{[a, b]} \sup f_i \cdot \delta}_{\leq U(\tilde{P}, f)} \\
\Rightarrow L(\tilde{P}^*, f) \geq L(P, f) \quad \leq U(\tilde{P}, f) \leq U(P, f)$$

$$L(P, f) - \sum \sup f_i \cdot \delta < S(\tilde{P}, f) < U(P, f) + \sum \sup f_i \cdot \delta$$

$$\int_a^b f - \varepsilon - \sum \sup f_i \cdot \delta < S(\tilde{P}, f) < \int_a^b f + \varepsilon + \sum \sup f_i \cdot \delta$$

$$\int_a^b f - 2\varepsilon < S(\tilde{P}, f) < \int_a^b f + 2\varepsilon$$

$$\Rightarrow \text{if } \|\tilde{P}\| < \delta, \text{ then } |S(\tilde{P}, f) - \int_a^b f| < 2\varepsilon$$

§ 6.3. Fundamental theorem of calculus.

thm Suppose $f \in R[a, b]$. and $F'(x) = f(x)$. $\forall x \in [a, b]$. then

$$\int_a^b f(x) dx = F(b) - F(a)$$

pf Since $f \in R[a, b]$. Given $\varepsilon > 0$, $\exists \delta > 0$ s.t. $|S(P, f) - \int_a^b f| < \varepsilon$ if $\|P\| < \delta$.

Let $P = \{x_0, \dots, x_n\}$ be a partition with $\|P\| < \delta$.

$$F(b) - F(a) = \sum_{i=1}^n F(x_i) - F(x_{i-1}) \stackrel{MT}{=} \sum f(t_i)(x_i - x_{i-1}) = S(P, f)$$

$$|F(b) - F(a) - \int_a^b f| < \varepsilon. \text{ Since } \varepsilon \text{ is arbitrary, } F(b) - F(a) = \int_a^b f.$$

thm Let $f \in R[a, b]$. Set $F(x) = \int_a^x f(t) dt$. $\forall x \in [a, b]$.

then F is cont. on $[a, b]$ & if f is cont. at c , $F'(c) = f(c)$.
 (Lipschitz).

pf) 1) Let $M = \sup_{[a,b]} |f|$.

$$\frac{y-x}{y-x} |F(y) - F(x)| = \left| \int_x^y f(t) dt \right| \leq \int_x^y |f(t)| dt \leq M(y-x).$$

$$|F(y) - F(x)| < M|y-x| \rightarrow \text{Lipschitz cont.}$$

$$\text{ii) } \frac{F(c+h) - F(c)}{h} = \frac{1}{h} \int_c^{c+h} f(t) dt.$$

$$\text{② } \frac{F(c+h) - F(c)}{h} - f(c) = \frac{1}{h} \int_c^{c+h} (f(t) - f(c)) dt.$$

given $\varepsilon > 0$, $\exists \delta > 0$ s.t. $|f(t) - f(c)| < \varepsilon$ if $|t - c| < \delta$, $t \in [a, b]$

since f is conti at c .

$$\left| \frac{F(c+h) - F(c)}{h} - f(c) \right| < \frac{1}{|h|} \int_c^{c+h} \varepsilon dt \text{ if } |h| < \delta. \quad \left(\frac{1}{h} \int_c^{c+h} f(t) dt \rightarrow f(c) \right)$$

$$= \varepsilon$$

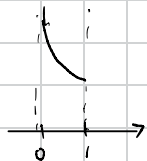
* change of variables $\rightarrow$ 2pt.

§ 6.4. Improper Riemann Integrals.

• $f: (0, 1] \rightarrow \mathbb{R}$.

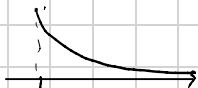
$$f(x) = x^{-1/2}$$

$\hookrightarrow$ upper sum $\rightarrow \infty$.



$\rightarrow$ not really closed interval partition, bounded interval not really.

• $f: [1, \infty) \rightarrow \mathbb{R}$.



$\rightarrow$ partition $\nabla \mathbb{R} \nrightarrow \mathbb{X}$.

Def $f: (a, b] \rightarrow \mathbb{R}$ & $f \in R[c, b] \quad \forall c \in (a, b]$. then define

$([a, b])$ or (a, b)

$$\int_a^b f(x) dx = \lim_{c \rightarrow a} \int_c^b f(x) dx \text{ provided that the limit exists.}$$

Let $f: [a, \infty) \rightarrow \mathbb{R}$ & $f \in R[a, c] \quad \forall c > a$. then define

$$\int_a^\infty f(x) dx = \lim_{c \rightarrow \infty} \int_a^c f(x) dx \text{ provided that the limit exists.}$$

thm Let $f, g: [a, \infty) \rightarrow \mathbb{R}$ & $f, g \in R[a, c]$. $\forall c > a$

Suppose $|f(x)| \leq g(x)$. $\forall x \geq a$ & $\int_a^\infty g(x) dx < +\infty$. * finite of $\int$ $\Rightarrow$ (bounded monotone) (exists & finite)

then $\int_a^\infty f(x) dx$ exists.

pf $u_n = \int_a^n f(x) dx$ $n \geq a$

$v_n = \int_a^n g(x) dx$ $n \geq a$.

$\hookrightarrow$ given $\varepsilon > 0$. $\exists p > 0$

$|\int_a^c f - L| < \varepsilon$ if $c > p$.

Since v_n converges. $\{v_n\}$ is a Cauchy seq. ($\rightarrow$ $\lim_{n \rightarrow \infty} v_n$ exists).

Let $n \geq m$. $|u_n - u_m| = |\int_m^n f(x) dx| \leq \int_m^n g(x) dx = v_n - v_m \rightarrow \textcircled{?}$

$\{u_n\}$ is a Cauchy seq. $\lim_{n \rightarrow \infty} \int_a^n f(x) dx = L$ exists.

Given $\varepsilon > 0$

Choose N such that $0 \leq \int_n^{n+1} g < \varepsilon/2$ if $n \geq N$.

$\hookrightarrow$ $\int_n^{n+1} g$ small.

& $|\int_a^n f - L| < \varepsilon/2$ if $n \geq N$.

$|\int_a^c f - L| \leq |\int_a^{[c]} f - L| + |\int_{[c]}^c f|$ if $c \geq N$.

$\leq |\int_a^{[c]} f - L| + \int_{[c]}^{[c]+1} g < \varepsilon/2 + \varepsilon/2$ if $c \geq N$.

$\hookrightarrow$ next step.

Ch 7. Series of Real Numbers

§ 7.1. Convergence Tests.

$\sum a_k$ converges iff $S_n = \sum_{k=1}^n a_k$ converges
(with partial sum)

positive series.

1) If $0 \leq d_k \leq L b_k$ for some $L > 0$, then

$\sum d_k$ converges if $\sum b_k$ converges. ($\Leftrightarrow \sum b_k$ diverges if $\sum a_k$ diverges)

2) $S_n = \sum_{k=1}^n a_k$ conv $\Leftrightarrow 0 \leq S_n \leq M$ for some M .

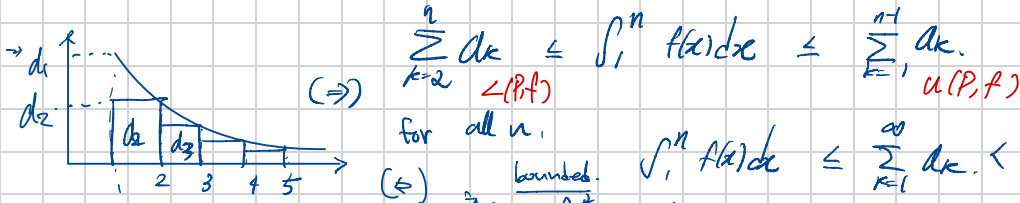
2) $\lim_{k \rightarrow \infty} \frac{dk}{bk} = L, L > 0$. Dann $\sum dk$ conv. iff $\sum bk$ conv.

$$\Rightarrow \frac{1}{2} \leq \frac{dk}{brc} \leq \frac{3}{2} \quad \forall k \geq N \text{ for some } N.$$
$$\sum d_k \text{ conv. off } \sum b_k \text{ conv. } (\text{if } 0 \neq \frac{1}{2}).$$

3) Let $f: [1, \infty) \rightarrow \mathbb{R}$ be integrable and decreasing fun.

$$\Rightarrow R[1, c], c > 1$$

Let $d_k = f(k)$ $k=1, 2, \dots$ then $\sum d_k$ conv. iff $\int_1^\infty f(x) dx < \infty$



for all n , $\int_1^n |f(x)| dx \leq \sum_{k=1}^{\infty} a_k < +\infty$.
 ($\Leftarrow$) bounded. $\int_1^x |f(x)| dx < \infty \quad \forall x \in \mathbb{R}_+^*$

4) $\lim_{n \rightarrow \infty} \frac{d\eta}{dn} < 1$
 > 1
 $= 1$

$\lim_{n \rightarrow \infty} \sqrt[n]{d\eta} < 1$
 > 1
 $= 1$

$\rightarrow$ increasing
 $\rightarrow$ 递增...

if $\sup_{n \rightarrow \infty} \sqrt[n]{a_n} < 1$ then $\sum a_n$ converges.

If $\limsup_{n \rightarrow \infty} \sqrt[n]{a_n} > 1$ then $\sum a_n$ diverges.

$\limsup_{n \rightarrow \infty} \sqrt[n]{a_n} = 1$ then undetermined.

$$\frac{1}{n} \log n^{-\alpha}$$

⑧ $\lim_{n \rightarrow \infty} \sqrt[n]{n} = 1$
 $\hookrightarrow$ $\frac{1}{\infty}$ $\frac{0}{0}$ $\frac{\infty}{\infty}$

pf A) Let $\sum_{n=1}^{\infty} \sqrt[n]{a_n} < p < 1$.

then $\sqrt[n]{a_n} < p$ if $n \geq N$ for some N .

$a_n < p^n$ if $n \geq N$.

Since $\sum p^n$ conv., $\sum a_n$ conv.

$\hookrightarrow$ ~~1/2, 1/3, 1/4, ...~~ $\sim N/n$
~~1/2, 1/3, 1/4, ...~~

B) choose $p > 0$ such that $1 < p < \lim_{n \rightarrow \infty} \sqrt[n]{a_n}$.

Then, there exists infinitely many $\sqrt[n]{a_n} > p$

For such n , $a_n \geq 1$. $\sum_{n=1}^{\infty} a_n = +\infty$.

$\S$ 17.2. the Dirichlet test.

$$\int_a^b fg = Fg|_a^b - \int_a^b Fg'$$

thm (Summation by Parts.) Let $1 \leq p \leq q$ Let $s_0 = 0$. ($s_n = \sum_{k=1}^n a_k$)

$$\sum_{k=p}^q a_k b_k = \sum_{k=p}^{q-1} s_k (b_k - b_{k+1}) + s_q b_q - s_{p-1} b_p$$

pf $a_k = s_k - s_{k+1}$ (driftung...)

thm Suppose $\sum_{k=1}^{\infty} a_k$

1) $|s_n| \leq M \quad \forall n$, for some M

2) $\{b_n\}$ is decreasing & $\lim_{n \rightarrow \infty} b_n = 0$.

then $\sum a_k b_k$ conv.

pf Given $\varepsilon > 0$, $\exists N$ s.t. $|\sum_{n=1}^q a_n b_n| < \varepsilon$ if $p, q \geq N$.

$$\sum_{k=p}^q a_k b_k = \sum_{k=p}^{q-1} s_k (b_k - b_{k+1}) + s_q b_q - s_{p-1} b_p$$

Given $\varepsilon > 0 \quad \exists N$ s.t. $|b_k| < \frac{\varepsilon}{1+M}$ if $k \geq N$.

Since $b_k \rightarrow 0$ as $k \rightarrow \infty$,

Ex1 $\sum_{n=1}^{\infty} (-1)^n a_n$ conv?

$\nexists a_n \rightarrow 0$

as $k \rightarrow \infty$.

Ex2 $\sum_{n=1}^{\infty} \frac{e^{i\theta n}}{n^{\alpha}}$ conv

if $\theta \neq 0$
($\theta \in [0, 2\pi)$)

max

$$\left| \sum_{k=p}^q a_k b_k \right| \leq \sum_{k=p}^{q+1} M(b_k - b_{k-1}) + 2M \frac{\varepsilon}{1+m} \text{ if } p, q \geq N+1$$

$$\leq M(b_p + b_{q+1}) + 2M \frac{\varepsilon}{1+m}$$

$$< 3\varepsilon \text{ if } p, q \geq N+1$$

Ex 2 $\sum \frac{\cos n\theta}{n^\alpha}, \sum \frac{\sin n\theta}{n^\alpha} \Rightarrow \sum \frac{e^{in\theta}}{n^\alpha} \quad (\alpha > 0, \theta \in [0, 2\pi])$

$$\sum_{k=1}^n e^{ik\theta} = \frac{e^{i\theta}(1 - e^{i(n+1)\theta})}{1 - e^{i\theta}} \quad (?)$$

$$\theta \neq 0, \left| \sum_{k=1}^n e^{ik\theta} \right| \leq \frac{\varepsilon}{(1 - \cos\theta)^2 - \sin^2\theta}^{1/2}$$

$$\sum e^{i\theta} = \cos\theta + i\sin\theta$$

§ 7.3. Absolute Convergence

Def We say $\sum_{k=1}^{\infty} a_k$ absolutely converges if $\sum_{k=1}^{\infty} |a_k|$ converges

thm $\hookrightarrow$ if $\sum_{k=1}^{\infty} a_k$ absolutely converges, $\sum_{k=1}^{\infty} a_k$ converges.

pf Given $\varepsilon > 0$, $\exists N$ s.t. $\left| \sum_{k=p}^q a_k \right| < \varepsilon$ if $p, q \geq N$ ($\Leftrightarrow S_n$ is Cauchy)

Let $\varepsilon > 0$, since $\sum_{k=1}^{\infty} |a_k| < \infty$, $\exists N$ s.t. $\sum_{k=p}^q |a_k| < \varepsilon$ if $p, q \geq N$.

$$\left| \sum_{k=p}^q a_k \right| \leq \sum_{k=p}^q |a_k| < \varepsilon \text{ if } p, q \geq N$$

Ex) $\sum (-1)^k \frac{1}{k}$ conv but not absolutely conv.

Def We say $\sum a_k$ conditionally conv. if $\sum a_k$ conv but $\sum |a_k|$ does not conv.

$\rightarrow$ 수열의 항의 절댓값의 합이 무한대인 경우. (absolute conv. & 수열)

Def Let $j: \mathbb{N} \rightarrow \mathbb{N}$ be a 1-1 & onto map, & $a'_k = a_{j(k)}$

$\sum a'_k$ is called a rearrangement of $\sum a_k$

thm $\star$ If $\sum a_k$ absolutely conv & $\sum a'_k$ is a rearrangement of $\sum a_k$, then $\sum_{k=1}^{\infty} a'_k = L$ where $\sum a_k = L$

$\hookrightarrow$ conditionally conv. $\rightarrow$ 수열의 항의 절댓값의 합이 무한대인 경우

Let $S_n = \sum_{k=1}^n a_k$. $S'_n = \sum_{k=1}^n a'_k$ Cauchy series 중 rearrange 되면 된다고 (양수일 때는)

Let $\varepsilon > 0$. Then, $\exists N$ s.t. $\sum_{k=N}^{\infty} |a_k| < \varepsilon/2$

$a'_k = a_{j(k)}$ for a 1-1 & onto map $j: \mathbb{N} \rightarrow \mathbb{N}$.

$p = \max j^{-1}(\{1, \dots, N\})$.

$\{1, \dots, N\} \subset \{j(1), \dots, j(p)\}$.

If $n \geq p$, $S_n - S'_n = \sum_{k=1}^n a_k - \sum_{k=1}^n a_{j(k)} \rightarrow (a_1 + a_2 + \dots + a_n) - (a_1 + a_2 + \dots + a_n)$

$|S_n - S'_n| = \sum_{k=N+1}^n |a_k| + \sum_{k=N+1}^n |a_k| < \varepsilon$

$S_n - \varepsilon < S'_n < S_n + \varepsilon$ if $n \geq p$. one arbitrary small number

$\underline{L} - \varepsilon \leq \underline{L} S'_n \leq \overline{L} S'_n \leq \overline{L} S_n + \varepsilon$

$\underline{L} - \varepsilon \leq \underline{L} S'_n \leq \overline{L} S'_n \leq \overline{L} S_n + \varepsilon$

$\underline{L} - \varepsilon \leq \underline{L} S'_n \leq \overline{L} S'_n \leq \overline{L} S_n + \varepsilon$

Since ε is arbitrary,

$\underline{L} S'_n = \overline{L} S'_n = L$

Thm Suppose $\sum a_k$ conditionally conv. then for every $\alpha \in \mathbb{R}$,

$\exists$ a rearrangement $\sum a'_k$ s.t. $\sum_{k=1}^{\infty} a'_k = \alpha$ (288x)

Ex) $(-1)^{k+1} \frac{1}{k} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \dots$

$\sum_{k=1}^{\infty} \frac{1}{2k-1} = \infty \Rightarrow \sum_{k=1}^{\infty} \frac{1}{2k} = \infty$

$2 > \sum_{k=1}^{N_1} a_k \geq 1$ of N_1

$2 > \sum_{k=N_1+1}^{N_2} a_k \geq 1$ of N_2 more

$2 > \sum_{k=N_1+1}^{N_2} a_k \geq 1$

$\underbrace{a_1 + \dots + a_{N_1}}_{\geq \frac{1}{2}} - \frac{1}{2} + \underbrace{a_{N_1+1} + \dots + a_{N_2}}_{\geq \frac{1}{2}} - \frac{1}{4} + \dots + a_{N_2+1} + \dots + a_{N_2} - \frac{1}{2} + \dots$
 $\geq (\frac{1}{2}) \cdot L \Rightarrow \sum a_k \rightarrow \infty$ / $(-\infty$ 쪽은 양의이 아니므로...)

Ch 8. Sequences & Series of fns.

§ 8.1. Pointwise Convergence

Let X be a metric space & $E \subset X$.

$f_n: E \rightarrow \mathbb{R}$ (real-valued fn seq.)

* ~~정확히~~ $f_n(x)$ 는 $f(x)$ 에 가까워진다 (점별 수렴)
 f_n 이 f 에 수렴.

Def. $\{f_n\}_{n=1}^{\infty}$ **conv. pointwise** in E if $\{f_n(x)\}_{n=1}^{\infty}$ conv. for every $x \in E$.

In this case, set $f(x) = \lim_{n \rightarrow \infty} f_n(x)$ & $f(x)$ is called

"the **pointwise limit** of $\{f_n\}$ "

* f_n **conti** $\rightarrow$ f **conti** ($\rightarrow$ ~~점별 수렴~~ $\rightarrow$ f 가 X 에 $\rightarrow$ f 가 X 에)

$\downarrow$ iff $\rightarrow$ $\downarrow$ iff

$\rightarrow$ integrable $\rightarrow$ integrable

$$f_n(x) = x^n, \quad x \in [0, 1] \rightarrow f(x) = \lim_{n \rightarrow \infty} f_n(x) = \begin{cases} 1 & x = 1 \\ 0 & x \in [0, 1) \end{cases}$$

$\Rightarrow$ 다 같은 'conv' 일지

§ 8.2. Uniform Convergence $\rightarrow$ conti, integ. 문제

$$\lim_{n \rightarrow \infty} \int_a^b f_n(x) dx \stackrel{?}{=} \int_a^b \lim_{n \rightarrow \infty} f_n(x) dx \quad \leftarrow \text{Lebesgue's theorem (dominant conv. thm)}$$

$$\lim_{n \rightarrow \infty} (f_n(x))' \stackrel{?}{=} \left(\lim_{n \rightarrow \infty} f_n(x) \right)' \quad \leftarrow \text{점별한지 알 수 없다. (uniform conv. 되면 OK)}$$

Def $f_n, f: E \rightarrow \mathbb{R}$.

We say $\{f_n\}$ converges **uniformly** to f on E if for every $\varepsilon > 0$,

$\exists N$ s.t. $|f_n(x) - f(x)| < \varepsilon \quad \forall x \in E \quad \text{if } n \geq N$.

(* ε 가 $\forall x \in E$ 에 대해 $\rightarrow$ pointwise conv.)

$\rightarrow$ point.

$\rightarrow$ ε 가 $\forall x \in E$ 에 대해

$$\Leftrightarrow \sup_{x \in E} |f_n(x) - f(x)| < \varepsilon \quad \text{if } n \geq N$$

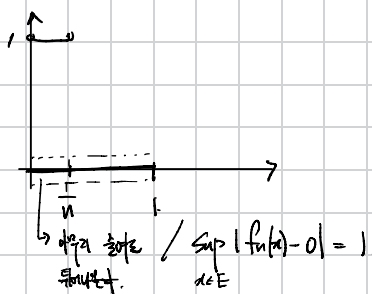
$\uparrow$
 ε 가 $\forall x \in E$ 에 대해

~~pointwise conv.~~
 $\rightarrow$ pointwise conv.

Ex) $f_n: (0,1) \rightarrow \mathbb{R}$.

$$f_n(x) = \begin{cases} 1 & \text{if } 0 < x < \frac{1}{n} \\ 0 & \text{o/w.} \end{cases}$$

$$\Rightarrow \lim_{n \rightarrow \infty} f_n(x) = 0 \quad \forall x \in (0,1).$$



Cauchy Criterion: $\lim_{n \rightarrow \infty} f_n(x) = f(x)$ is Cauchy.

Thm $\{f_n\}$ uniformly converges on E iff $\{f_n\}$ is Cauchy for every $\varepsilon > 0$.

$\exists N$ such that $|f_n(x) - f_m(x)| < \varepsilon$ $\forall x \in E$ if $n, m \geq N$.

$\Rightarrow$ If $\{f_n\}$ satisfies (*), then $\{f_n\}$ is uniformly Cauchy.

Pf $(\Rightarrow) \exists f: E \rightarrow \mathbb{R}$ s.t. $f(x) = \lim_{n \rightarrow \infty} f_n(x)$.

Given $\varepsilon > 0$, $\exists N > 0$ s.t. $|f_n(x) - f(x)| < \varepsilon/2$ $\forall x \in E$ if $n \geq N$.

$|f_m(x) - f(x)| < \varepsilon/2$ $\forall x \in E$ if $m \geq N$.

$|f_n(x) - f_m(x)| \leq |f_n(x) - f(x)| + |f(x) - f_m(x)| < \varepsilon/2 + \varepsilon/2 = \varepsilon$ $\forall x \in E$ if $n, m \geq N$.

$(\Leftarrow) \{f_n(x)\}$ is Cauchy for all $x \in E$.

$\exists \lim_{n \rightarrow \infty} f_n(x)$. Set $f(x) = \lim_{n \rightarrow \infty} f_n(x)$.

Given $\varepsilon > 0$, $\exists N$ s.t. $|f_n(x) - f_m(x)| < \varepsilon$ $\forall x \in E$ if $n, m \geq N$.

Letting $m \rightarrow \infty$, $|f_n(x) - f(x)| \leq \varepsilon$ $\forall x \in E$ if $n \geq N$.

$\Leftrightarrow \sup_{x \in E} |f_n(x) - f_m(x)| < \varepsilon$.

Weierstrass M-test (E8.8!)

thm Suppose $f_n: E \rightarrow \mathbb{R}$ satisfies $|f_n(x)| \leq M_n \quad \forall x \in E$. for some $M_n \geq 0$

if $\sum_{n=1}^{\infty} M_n < +\infty$, then $\sum_{n=1}^{\infty} f_n(x)$ uniformly conv. on E . not ~~the~~ bound

pf $S_n(x) = \sum_{k=1}^n f_k(x)$

Since $\sum M_k < +\infty$, given $\varepsilon > 0$. $\exists N$ st. $\sum_{k=p}^{\infty} M_k < \varepsilon$ if $p, q \geq N$.

Let $n > m$, $|f_n(x) - f_m(x)| \leq \sum_{k=m+1}^n |f_k(x)| \leq \sum_{k=m+1}^n M_k < \varepsilon, \quad \forall x \in E$ if $n, m \geq N$.

Ex $\sum_{n=1}^{\infty} \frac{1}{n^2} x^n \rightarrow \frac{1}{n} \lim_{n \rightarrow \infty} \sqrt[n]{\frac{1}{n^2}}$ if $|x| \leq r < R$. $\sum_{n=1}^{\infty} \frac{1}{n^2} x^n$ conv. on $\{|x| \leq r\}$ ①

$\sum_{n=1}^{\infty} \frac{\cos nx}{n^2}$ uniformly conv on $\mathbb{R}$ $\because |\frac{\cos nx}{n^2}| \leq \frac{1}{n^2}$ & $\sum \frac{1}{n^2} < +\infty$

§ 8.7. Uniform Convergence & Continuity

thm Let $f_n: E \rightarrow \mathbb{R}$ and $f_n \rightarrow f$ on E suppose p is a limit point of E & $(p \in E \text{ or } p \notin E)$
(unif. conv.)

$\lim_{x \rightarrow p} f_n(x) = A_n$ Then, $A_n \rightarrow A$ conv. & $\lim_{x \rightarrow p} f(x) = \lim_{n \rightarrow \infty} A_n = A$ ($\Leftrightarrow \lim_{x \rightarrow p} \lim_{n \rightarrow \infty} f_n(x) = \lim_{n \rightarrow \infty} \lim_{x \rightarrow p} f_n(x)$)

pf ① Given $\varepsilon > 0$, $\exists N$ st. $|f_n(x) - f_m(x)| < \varepsilon, \quad \forall x \in E$ if $n, m \geq N$ (unif. conv. $\rightarrow$ unif. cont.)

Since $f_n \rightarrow f$

Let $n > p$. $|A_n - A_m| \leq \varepsilon$ if $n, m \geq N$.

$\{A_n\}$ is a Cauchy seq. A_n conv. Set $A = \lim_{n \rightarrow \infty} A_n$

② (need to show $\forall \varepsilon > 0, \exists \delta > 0$ st. $|f(x) - A| < \varepsilon$ if $0 < d(x, p) < \delta$ & $x \in E$)

$f(x) - A = f(x) - f_n(x) + f_n(x) - A_n + A_n - A$ ($|f_n(x) - f(x)| < \varepsilon/3 \quad \forall x \in E$ if $n \geq N$)

(why) Given $\varepsilon > 0, \exists N$ st. $|f_n(x) - f(x)| < \varepsilon/3 \quad \forall x \in E$.

$|A_n - A| < \varepsilon/3$

Since $\lim_{x \rightarrow p} f_n(x) = A_n, \exists \delta > 0$ st. $|f_n(x) - A_n| < \varepsilon/3$ if $0 < d(x, p) < \delta, x \in E$.

if $0 < d(x, p) < \delta, x \in E, |f(x) - A| \leq |f(x) - f_n(x)| + |f_n(x) - A_n| + |A_n - A| < \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon$.

Cor^{*} If $f_n: E \rightarrow \mathbb{R}$ is continuous & $f_n \rightarrow f$ on E , then f is conti on E .

pf Let $p \in E$. Since $f_n \rightarrow f$ on E ,

$$\lim_{x \rightarrow p} f(x) = \lim_{n \rightarrow \infty} \lim_{x \rightarrow p} f_n(x) = \lim_{n \rightarrow \infty} f_n(p) = f(p) \quad \therefore f \text{ is conti on } p.$$

Ex^{*} $f_n(x) = x^n \quad x \in [0, 1]$. $\lim_{n \rightarrow \infty} f_n(x) = f(x)$.

$$f(x) = \begin{cases} 1 & \text{if } x=1 \\ 0 & \text{if } 0 \leq x < 1 \end{cases} \rightarrow \begin{array}{c} \bullet \\ | \\ \bullet \end{array} \rightarrow \lim_{n \rightarrow \infty} \lim_{x \rightarrow 1} f_n(x) = 1 + \lim_{x \rightarrow 1} \lim_{n \rightarrow \infty} f_n(x) = 0.$$

$\hookrightarrow$ not ctt...

$C(K)$ space

Def Let K be a cpe subset of a metric space X .

$$C(K) = \{ f: K \rightarrow \mathbb{R} : f \text{ is continuous} \}.$$

$$\text{Let } f \in C(K), \quad \|f\|_u = \max_{x \in K} |f(x)|.$$

$$\text{For } f, g \in C(K), \quad d(f, g) = \|f - g\|_u = \max_{x \in K} |f(x) - g(x)|$$

thm^{*} Then $(C(K), d)$ is a complete metric space.

pf Let $\{f_n\}$ be a Cauchy seq. in $C(K)$.

$$\text{Given } \varepsilon > 0, \exists N \text{ s.t. } |f_n(x) - f_m(x)| < \varepsilon \quad \forall x \in K \text{ if } n, m \geq N.$$

$$\Leftrightarrow \max_{x \in K} |f_n(x) - f_m(x)| < \varepsilon \quad \text{if } n, m \geq N$$

$$\Leftrightarrow d(f_n, f_m) < \varepsilon \quad \text{if } n, m \geq N.$$

$\{f_n\}$ is uniformly Cauchy on K , $\therefore f_n(x)$ conv on K .

$$\text{See } f(x) = \lim_{n \rightarrow \infty} f_n(x), \quad \underline{f_n \rightarrow f \text{ on } K}.$$

By Corollary, $f \in C(K)$ $\therefore C(K)$ is complete metric space.

at least 3
at least 2

§8.4 Uniform Convergence & Integration

Thm Let $f_n \in R[a, b]$ & $f_n \rightarrow f$ on $[a, b]$. Then $f \in R[a, b]$ &

$$\lim_{n \rightarrow \infty} \int_a^b f_n(x) dx = \int_a^b f(x) dx \quad \left(\lim_{n \rightarrow \infty} \int_a^b f_n(x) dx = \int_a^b \lim_{n \rightarrow \infty} f_n(x) dx \right)$$

pf Since $f_n \rightarrow f$ on $[a, b]$, let $\varepsilon > 0$ $\exists N$ st. $|f_n(x) - f(x)| < \varepsilon \quad \forall x \in [a, b]$
if $n \geq N$.

$$f_n(x) - \varepsilon < f(x) < f_n(x) + \varepsilon$$

$$\int_a^b f_n - \varepsilon(b-a) = \int_a^b f \leq \int_a^b f \leq \int_a^b (f_n + \varepsilon) = \int_a^b f_n + \varepsilon(b-a)$$

$$0 \leq \int_a^b f - \int_a^b f_n \leq \varepsilon(b-a). \quad \text{Let } \varepsilon \rightarrow 0. \quad \int_a^b f = \int_a^b f_n \quad \therefore f \in R[a, b].$$

$$|\int_a^b f - \int_a^b f_n| \leq \int_a^b |f - f_n| \leq \varepsilon(b-a). \quad \text{if } n \geq N. \quad \therefore \int_a^b f_n \rightarrow \int_a^b f \text{ as } n \rightarrow \infty$$

②

§8.5 Uniform Convergence & Differentiation

Thm Suppose f_n are differentiable on $[a, b]$. If

a) f_n' uniformly conv. on $[a, b]$

b) $f_n(x_0)$ conv for some $x_0 \in [a, b]$. Then

$$f_n \rightarrow f \text{ for a func } f \text{ on } [a, b] \quad \& \quad f'(x) = \lim_{n \rightarrow \infty} f_n'(x).$$

* If we additionally assume f_n' is contin on $\left(\lim_{n \rightarrow \infty} f_n(x)\right)$
on $[a, b]$

$$\text{pf } f_n(x) = \int_{x_0}^x f_n'(t) dt + f_n(x_0) \quad \text{Let } f_n' \rightarrow h \quad \& \quad \lim_{n \rightarrow \infty} f_n(x) = a$$

$$f(x) = \lim_{n \rightarrow \infty} f_n(x) = \int_{x_0}^x h(t) dt + a \quad \rightarrow \text{left \& cont}$$

$$\text{wrt } \downarrow \quad f'(x) = h(x) = \lim_{n \rightarrow \infty} f_n'(x).$$

pf Since f'_n uniformly conv. on $[a, b]$

given $\varepsilon > 0$. $\exists N$ s.t. $|f'_n(x) - f'_m(x)| < \varepsilon$, $\forall x \in [a, b]$ if $n, m \geq N$.

[observation] Let $x, y \in [a, b]$

$$(f'_n(x) - f'_n(y)) - (f'_m(x) - f'_m(y)) = (f'_n(x) - f'_m(x)) - (f'_n(y) - f'_m(y))$$

$$= \underbrace{(f'_n(t) - f'_m(t))}_{\text{error}} (x-y) \text{ for some } t \in (x, y)$$

$$|(f'_n(x) - f'_n(y)) - (f'_m(x) - f'_m(y))| \leq \underline{\varepsilon} (x-y) \text{ for } \forall x, y \in [a, b] \text{ if } n, m \geq N. (*)$$

$$f_n(x) - f_m(x) = f_n(b) - f_n(a_0) + f_n(a_0) - f_m(a_0) + f_m(a_0) - f_m(x)$$

$$|f_n(x) - f_m(x)| \leq |(f_n(x) - f_n(a_0)) - (f_m(x) - f_m(a_0))| + \underbrace{|f_n(a_0) - f_m(a_0)|}_{\text{max}}$$

Taking $y = a_0$ in $(*)$,

$$|f_n(x) - f_m(x)| \leq \varepsilon (b-a) + \varepsilon \quad \forall x \in [a, b] \text{ if } n, m \geq N.$$

$\hookrightarrow$ choose N s.t.

$$|f_n(a_0) - f_m(a_0)| < \varepsilon$$

$\Rightarrow f'_n$ uniformly Cauchy in $[a, b]$. $\hookrightarrow$ ok

$$\text{Set } f(x) = \lim_{n \rightarrow \infty} f_n(x)$$

Fix $p \in [a, b]$

$$g_n(t) = \frac{f_n(t) - f_n(p)}{t-p}, \quad g(t) = \frac{f(t) - f(p)}{t-p}, \quad t \neq p.$$

claim: $\lim_{n \rightarrow \infty} \lim_{t \rightarrow p} g_n(t) = \lim_{t \rightarrow p} g(t)$.

(*) or let $| \frac{f_n(t) - f_n(p)}{t-p} - \frac{f_m(t) - f_m(p)}{t-p} | < \varepsilon, \quad \forall t \neq p \in [a, b] \text{ if } n, m \geq N.$

$$|g_n(t) - g_m(t)| < \varepsilon \quad \forall t \neq p \text{ if } n, m \geq N.$$

$\Rightarrow g_n$ is uniformly Cauchy (conv) $\nearrow g_n \rightarrow g$ on $[a, b] \setminus \{p\}$.

$m_2 \neq m_3$ $\nearrow$ $\therefore$ letting $m \rightarrow \infty$,
 $|g_n(t) - g(t)| \leq \varepsilon \quad \forall t \neq p \text{ if } n \geq N$

$$\lim_{n \rightarrow \infty} \lim_{t \rightarrow p} g_n(t) = \lim_{t \rightarrow p} \lim_{n \rightarrow \infty} g_n(t) = \lim_{t \rightarrow p} g(t).$$

11.29.4

§ 8.6. The Weierstrass Approximation Theorem

Thm Let $f: [a, b] \rightarrow \mathbb{R}$ be continuous. Then for every $\varepsilon > 0$, $\exists$ a polynomial P

$$\text{st. } |f(x) - P(x)| < \varepsilon, \quad \forall x \in [a, b].$$

$\Rightarrow \max_{x \in [a, b]} |f(x) - P(x)| = d(f, p)$

$$\Leftrightarrow f \in C([a, b]) \Rightarrow \forall \varepsilon > 0 \exists \text{ a poly } P \text{ st. } d(f, p) < \varepsilon$$

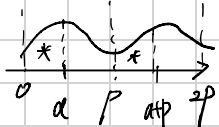
$\Leftrightarrow$ the set of polynomial is a dense subset of $(C([a, b]), d)$

* If f is analytic at $c \in (a, b)$, $f(x) = \sum_{k=0}^{\infty} f^{(k)}(c) \frac{(x-c)^k}{k!} \Rightarrow$ ^② Taylor's thm

pf for $f \in C([a, b])$, $\exists$ a seq of poly $\{P_n\}$ st. $P_n \rightarrow f$ on $[a, b]$ _{uniformly conv} $\Rightarrow$ W.T.T.

Def We say f is p -periodic (a periodic fun with period p) if $f(x+p) = f(x) \quad \forall x \in \mathbb{R}$

thm If $f \in R[0, p]$ & f is p -periodic, then $f \in R[a, a+p]$ & $\int_a^{a+p} f(x) dx = \int_0^p f(x) dx$

$\Rightarrow$  $\begin{matrix} \text{i) } f \in R[a, p] \text{ \& } f \in R[p, a+p] \\ \text{ii) } \text{the area} \end{matrix}$

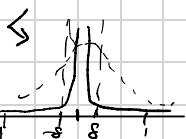
* Approximation to the Identity (Approximate Identity)

Def We say $\{Q_n\}$ is an approximation to the identity if

① nonnegative $Q_n \in R[1, 1]$ satisfies

1) $\int_1^1 Q_n(t) dt = 1$

2) For $\delta > 0$, $\int_{\delta \leq |t| \leq 1} Q_n(t) dt \rightarrow 0$ as $n \rightarrow \infty$



$\rightarrow$ approx to the identity
 special case

* if Q_n is not nonnegative
 $\int_{\delta \leq |t| \leq 1} |Q_n(t)| dt \rightarrow 0$ as $n \rightarrow \infty$

Then suppose $\{Q_n\}$ is an approximation to the identity & $f \in R[1,1]$ is

2-periodic. For $n \in \mathbb{N}$, $S_n(x) = \int_{-1}^1 f(x+t) Q_n(t) dt$. /* delta func approx? */

A) If f is continuous at x , $\lim_{n \rightarrow \infty} S_n(x) = f(x)$. /* f is continuous at x .
 - identity */

B) If f is continuous on $[1,1]$, $S_n \rightarrow f$ on $[1,1]$.
($\hookrightarrow$ uniformly conti. & periodic def f is uniformly conti on $\mathbb{R}$).

* S_n is 2-periodic. $\Rightarrow S_n \rightarrow f$ on $\mathbb{R}$. ($S_n(x+2) = \int_{-1}^1 f(x+2+t) Q_n(t) dt = \int_{-1}^1 f(x+t) Q_n(t) dt = S_n(x)$).

pf A) $S_n(x) - f(x) = \int_{-1}^1 f(x+t) Q_n(t) dt - \underbrace{f(x) \int_{-1}^1 Q_n(t) dt}_{=1} \quad \textcircled{2}$
 $= \int_{-1}^1 [f(x+t) - f(x)] Q_n(t) dt.$ /* $\Rightarrow$ approx zero. */

Since f is conti at x , given $\varepsilon > 0$, $\exists \delta > 0$ s.t. $|f(x+t) - f(x)| < \varepsilon$ if $|t| < \delta(x)$.
 $\delta \in (0,1)$

(*) $|S_n(x) - f(x)| \leq \underbrace{\int_{|t| < \delta} |f(x+t) - f(x)| Q_n(t) dt}_{<1} + \int_{\delta \leq |t| \leq 1} |f(x+t) - f(x)| Q_n(t) dt.$ /* δ is $\delta(x)$ */

($f \in R$ -integrable def bounded) $\Rightarrow$ $\textcircled{2}$

$|S_n(x) - f(x)| \leq \varepsilon + 2 \sup_{x \in \mathbb{R}} |f| \int_{\delta \leq |t| \leq 1} Q_n(t) dt.$

$\lim_{n \rightarrow \infty} |S_n(x) - f(x)| \leq \varepsilon + 2 \sup_{x \in \mathbb{R}} |f| \cdot \lim_{n \rightarrow \infty} \int_{\delta \leq |t| \leq 1} Q_n(t) dt$
($\Rightarrow 0$ by 2.1.17)

$\lim_{n \rightarrow \infty} |S_n(x) - f(x)| \leq \varepsilon$, taking $\varepsilon \rightarrow 0$ $\lim_{n \rightarrow \infty} |S_n(x) - f(x)| = 0$,
 $\lim_{n \rightarrow \infty} |S_n(x) - f(x)| = 0$.

$\forall \varepsilon, \forall \delta > 0 \exists N$ s.t. $|S_n(x) - f(x)| < \varepsilon$.

$\Rightarrow$ choose N s.t. $\int_{\delta \leq |t| \leq 1} Q_n(t) dt \leq \frac{\varepsilon}{2 \sup |f| + 1}$ if $n \geq N$.

B) Given $\varepsilon > 0$, $\exists \delta > 0$ s.t. $|f(x+t) - f(x)| < \varepsilon$ if $|t| < \delta$ independent of x

$$(*) \rightarrow \| \zeta_n(t) - f(t) \| \leq \int_{|t| \leq \delta} |f(t+\epsilon) - f(t)| Q_n(t) dt + 2 \sup_{|t| \leq R} |f_t| \int_{\delta \leq |t| \leq 1} Q_n(t) dt.$$

$$\leq \varepsilon \cdot \int_{|H| \leq \delta} |u(t)| dt + 2 \sup_{x \in \mathbb{R}} |A| \int_{|x| \geq \delta} |u(t)| dx, \quad \underline{x \in \mathbb{R}}.$$

$$\sup_{x \in \mathbb{R}} |f_n(x) - f(x)| \leq \varepsilon + 2 \sup_{t \in \mathbb{R}} |f(t)| \int_{|x| \geq \frac{1}{\varepsilon}} |g(t)| dt.$$

$$\forall \epsilon > 0 \quad \sup_{x \in \mathbb{R}} |f_n(x) - f(x)| \leq \epsilon$$

$$\lim_{n \rightarrow \infty} \sup |g_n(x) - f(x)| = 0$$

$$\therefore S_n \ni f \text{ on } \mathbb{R}.$$

$\Rightarrow 12 \text{ bits!}$

pf. d. Wertgrössen ~~approx.~~ thun.

We only need to consider $[0, 1]$ for this.

pf $g(x) = f((b-a)x + a)$. $g \in C([0,1]) \rightarrow$ get interval poly ~~not~~ d.

$$\therefore |g-p| = |f(ba) + a - p(a)| < \varepsilon \Rightarrow |f(y) - p(\frac{y-a}{ba})| < \varepsilon.$$

예제. $g(x) = \frac{g(0) + (g(1) - g(0))x}{\text{poly } 1/K_2}$ $\rightarrow$ $1/2$ approx $\frac{1}{K_2}$ poly $\frac{1}{K_2}$ $\frac{1}{K_2}$ (poly $\frac{1}{K_2}$ $\frac{1}{K_2}$).

$$\rightarrow g(1)=0, \quad g(0)=0.$$



$\Rightarrow$ " for $f \in C([0,1])$ & 2-periodic satisfying $f=0$ on $[1,2]$,

$\exists$ a seq of poly P_n st. $P_n \rightarrow f$ on $[0,1]^d$

For $x \in [0, 1]$

$$S_n(d) = \int_{-1}^1 f(d+t) Q_n(t) dt = \int_{\substack{-1 \\ d \text{ stationen}}}^{\substack{1 \\ d \text{ stationen}}} f(d+t) Q_n(t) dt = \int_0^1 f'(x) \underbrace{Q_n(t-d)}_{\substack{\text{stationen} \\ \text{stationen}}} dt$$

We choose $Q_n(t) = \ln(1-t^2)^n$ s.t.

$$1) \int_1^{\infty} \frac{1}{t^2} dt = 1 \quad (\rightarrow \text{Ca} \int_1^{\infty} (1-t^2)^n dt = 1).$$

2) $\forall \epsilon > 0, \int_{\|f\| \leq 1} Q_n(f) df \rightarrow 0$ as $n \rightarrow \infty$

1) $C_n^{-1} = \int_{-1}^1 (1-t^2)^n dt = 2 \int_0^1 (1-t^2)^n dt \geq 2 \int_0^{\frac{\sqrt{n}}{2}} (1-t^2)^n dt \geq 2 \int_0^{\frac{\sqrt{n}}{2}} (1-nt) dt$
 $\Rightarrow C_n \leq \frac{3}{4}\sqrt{n}$ $\begin{matrix} \uparrow \\ (1-x)^n \geq 1-nx \end{matrix} = \frac{4}{3\sqrt{n}}$

2) $\int_{|x| \leq 1} \underbrace{C_n(1-t^2)^n}_{\text{decreases}} dt \leq 2 \int_0^1 \frac{3}{4}\sqrt{n} (1-t^2)^n dt$
 $\leq \frac{3}{2}\sqrt{n} (1-t^2)^n \rightarrow 0 \text{ as } n \rightarrow \infty$

By thm, $\underbrace{C_n \rightarrow f}_{\text{uniformly conv. on } \mathbb{R}}$

(*) $\rightarrow \int_0^1 f(t) \cdot \underbrace{C_n(1-(t-x)^2)^n}_{\rightarrow \sum c_n t^k dx} dt \rightarrow \text{if } 0 \leq x \leq 1, \underbrace{C_n(x)}_{\text{(nonnegative)}} \text{ is a polynomial.}$
 $\therefore \underline{C_n(x)} \rightarrow f \text{ on } [0, 1].$

* continuous for $\rightarrow$ approx. by poly. all.

$\rightarrow$ $\forall \epsilon > 0$ $\exists \delta > 0$ $\exists N \in \mathbb{N}$ s.t.

$\rightarrow$ $\forall n > N$ $\forall x \in [0, 1]$ $|C_n(x) - f(x)| < \epsilon$ (i.e. $C_n \rightarrow f$ uniformly on $[0, 1]$)