

Ch 1.

§ 1.1. Sets and Operations on Sets

def (집합) 정해진 f 있는 것들의 모임

def (Power Set) Let A is a Set.

$$P(A) = \text{Set of all subsets of } A \quad (2^A)$$

def (Cartesian Product) $A \times B = \{ (a, b) : a \in A, b \in B \}$

§ 1.2. Functions

def $f: A \rightarrow B$ on mapping.

A : domain of f (정의역), Dom f

B : range of f (값역). Range f

* if range of $f = B$, f is onto (전사함수)

* if $a_1 = a_2$ whenever $f(a_1) = f(a_2)$, f is one to one (1-1)

def (graph of f) $= \{ (a, f(a)) \in A \times B : a \in A \}$.

def Let $E \subset A$, (image of E) $= \{ f(a) : a \in E \} = f(E)$

Let $H \subset B$, (preimage of B) $= \{ a \in A : f(a) \in H \} = f^{-1}(H)$

prop. a) $f^{-1}(f(E)) \supset E$.

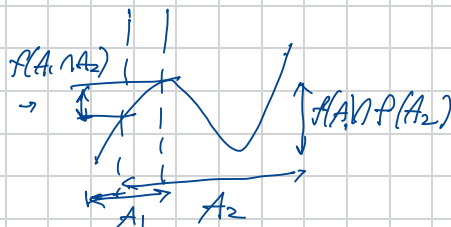
b) $f(f^{-1}(H)) \subset H$.

Thm Let $A_1, A_2 \subset A$. then,

a) $f(A_1 \cup A_2) = f(A_1) \cup f(A_2)$

b) $f(A_1 \cap A_2) \subset f(A_1) \cap f(A_2)$

* Ex 1 b)



Let $B_1, B_2 \subset B$. then,

c) $f^{-1}(B_1 \cup B_2) = f^{-1}(B_1) \cup f^{-1}(B_2)$

* d) $f^{-1}(B_1 \cap B_2) = f^{-1}(B_1) \cap f^{-1}(B_2)$

e) $f^{-1}(B^c) = (f^{-1}(B))^c$

($\Rightarrow$)

pf of d) i) $B_1 \cap B_2 \subset B_1, B_1 \cap B_2 \subset B_2$ o/e/z

$$f^{-1}(B_1 \cap B_2) \subset f^{-1}(B_1) \cap f^{-1}(B_2)$$

($\Leftarrow$)

ii) Let $a \in f^{-1}(B_1) \cap f^{-1}(B_2)$

$$f(a) \in B_1 \text{ and } f(a) \in B_2 \text{ o/e/z } f(a) \in B_1 \cap B_2$$

$$a \in f^{-1}(B_1 \cap B_2) \text{ o/e/z}$$

$$f^{-1}(B_1) \cap f^{-1}(B_2) \subset f^{-1}(B_1 \cap B_2)$$

§ 1.3 Mathematical Induction

Thm (Principle of Mathematical Induction)

for $n \in \mathbb{N}$, Let $P(n)$ be a statement. Suppose

a) $P(1)$ is true

b) $P(n+1)$ is true whenever $P(n)$ is true

then, $P(n)$ is true for all $n \in \mathbb{N}$.

Def = Well-ordered Principle of $\mathbb{N}$

: every non-empty subset of $\mathbb{N}$ has the smallest element

pf wop $\rightarrow$ PMI. <by contradiction> - on 201 wop + 202?

Suppose the conclusion is false. That is to say,

$\exists k_0 \in \mathbb{N}, k_0 \geq 1$, s.t. $P(k_0)$ is false

Let $A = \{k \in \mathbb{N} : P(k) \text{ is false}\}$. $A \neq \emptyset$

Since $A \subset \mathbb{N}$, by wop there is the smallest element $n_{(z2)}$ of A .

$P(n)$ is false and $P(n+1)$ is true, this contradicts b) ~~is~~

§ 1.4. Least Upper Bound Property.

체의 성질

1) $a, b \in F$ 일때 $a+b \in F, a \cdot b \in F$

2) (교환법칙) $a+b = b+a, a \cdot b = b \cdot a$

3) (결합법칙) $a+(b+c) = (a+b)+c, a \cdot (b \cdot c) = (a \cdot b) \cdot c$

4) (덧셈의 항등원) $\exists 0 \in F$ s.t. $a+0 = a$

5) (덧셈의 역원) $\forall a \in F, \exists -a \in F$ s.t. $a+(-a) = 0$

6) (곱셈의 항등원) $\exists 1 \in F$ s.t. $a \cdot 1 = a$

7) (곱셈의 역원) $\forall a \in F, \exists a^{-1} \in F$ s.t. $a \cdot a^{-1} = 1$

8) (분배법칙) $a \cdot (b+c) = ab+ac$

체의 예시: $\mathbb{Q}, \mathbb{R}, \{r_1+r_2\sqrt{2} : r_1, r_2 \in \mathbb{Q}\}$

def (Positive Real Set) $\exists P \subset \mathbb{R}$ s.t.

- "ordered properties" [
- 1) if $a, b \in P$, $a-b \in P$ & $a+b \in P$
 - 2) if $a \in \mathbb{R}$, only one of the following holds.
: $a \in P$, $-a \in P$, $a=0$

def Let $a, b \in \mathbb{R}$, we say $a > b$ if $a-b \in P$.

~~def~~ $a \in P \Leftrightarrow \underbrace{a-0 \in P} \Leftrightarrow a > 0$

Properties of $>$

1) $a, b \in \mathbb{R}$, $a > b$, $a=b$, $a < b$ $\exists$ at least one.

1) if $a > b$, $a+c > b+c$

2) if $a > b$ & $c > 0$, $a \cdot c > b \cdot c$

3) if $a > b$ & $c < 0$, $a \cdot c < b \cdot c$

4) if $a \neq 0$, $a^2 = a \cdot a > 0$

5) if $a > 0$, $\frac{1}{a} > 0$ if $a < 0$, $\frac{1}{a} < 0$.

pf 2) $a-b \in P$, $c \in P \Rightarrow (a-b) \cdot c \in P$.

def (Upper Bound of a Set) Let $E \subset \mathbb{R}$ & $E \neq \emptyset$. We say

E is "bounded above" if there is $b \in \mathbb{R}$

s.t. $x \leq b$, $\forall x \in E$.

In such a case, b is called an upper bound of E .

Example A $E = \left\{ \frac{n}{n+1}; n = 0, 1, 2, \dots \right\}$

- 1) 2, 100 are upper bounds of E .
- 2) any element of E cannot be an upper bound of E .
- 3) 1 is an upper bound of E .
if $b < 1$, b is not an upper bound.

Ex B $\mathbb{N} = \{1, 2, \dots\}$ has no upper bound.

Ex C $B = \{r \in \mathbb{Q} : r > 0, r^2 < 2\}$

- 1) 2 is an upper bound of B
- 2) B does not have an upper bound in itself.

↳ Let $b \in B$. $\sqrt{2}$ is an upper bound of B

$$\tilde{b} = b + \frac{2-b^2}{2+b} < \sqrt{2} \quad \nmid$$

$$b < \tilde{b} < \sqrt{2}, \quad \tilde{b}^2 < 2 \quad \text{or} \quad \tilde{b} \in B$$

def (Least Upper Bound) Let $E \subset \mathbb{R}$ & $E \neq \emptyset$

An element α of E is called "least upper bound of E " if

- 1) α is an upper bound of E
- 2) if $\beta < \alpha$, β is not an upper bound of E .
($\Leftrightarrow$ if β is an upper bound of E , $\beta \geq \alpha$)

by 2), the least upper bound of a set is unique. $\left. \begin{matrix} a_1 \geq a_2 \\ a_1 \leq a_2 \end{matrix} \right\} \Rightarrow a_1 = a_2$

We denote $\alpha = \sup E$ (supremum of E)

상한.

Thm 1.4.4. Let $E \subset \mathbb{R}$ & $E \neq \emptyset$, suppose E has an upper bound $\alpha \in \mathbb{R}$.

then, $\alpha = \sup E$, iff for every $r < \alpha$, there is an $x \in E$ s.t. $r < x (\leq \alpha)$

pf ($\Rightarrow$) Since α is an upper bound, it is enough to show $\Leftarrow$

Let $\beta < \alpha$, $\exists x \in E$ s.t. $\beta < x$

so, β is not an upper bound.

($\Rightarrow$) $\alpha = \sup E$ & $r < \alpha$ is not an upper bound of E .

so $\exists x \in E$ s.t. $r < x < \alpha$.

$\forall \varepsilon$ statement: for $\forall \varepsilon > 0$, there is an $x \in E$ s.t. $\sup E - \varepsilon < x$

Ex $\sup \left\{ \frac{n}{n+1} : n=1, 2, \dots \right\} = 1$.

$\sup \{ r \in \mathbb{Q} : r > 0, r^2 < 2 \} = \sqrt{2}$

def (Least Upper Bound Property of $\mathbb{R}$).

Every nonempty subset of $\mathbb{R}$ that is bounded above

has a supremum (least upper bound) in $\mathbb{R}$

$\Rightarrow$ "Completeness of $\mathbb{R}$ "

def Let $E \subset \mathbb{R}$ & $E \neq \emptyset$, then

$\sup E = -\infty$ if E is not bounded above

$\inf E = \infty$ if E is not bounded below.

§1.4. Consequences of Least Upper Bound Property

Thm 1.5.1. (Archimedean Property) Let $y \in \mathbb{R}$, $x > 0$, then $\exists n \in \mathbb{N}$ s.t.

$$nx > y$$

$\Leftrightarrow \mathbb{N}$ is not bounded

$$\Leftrightarrow \forall \varepsilon > 0, \exists n \in \mathbb{N} \text{ s.t. } n\varepsilon > 1 \quad (\varepsilon > \frac{1}{n})$$

pf $y \leq 0$ is not to prove.

$y > 0$ is. Suppose not for some $y \in \mathbb{R}$, there is no $n \in \mathbb{N}$ s.t.

$$nx > y \quad (\text{i.e. } \underbrace{nx \leq y}_{\downarrow} \text{ for } \forall n \in \mathbb{N}).$$

$E = \{nx : n \in \mathbb{N}\}$ is bounded above

$E \neq \emptyset$, $\sup E$ exists. ← B.1.1.2.1

$$\sup E - x < n_0 x$$

$$\sup E < (n_0 + 1)x \in E \quad \text{contradiction}$$

Thm 1.5.2. ^(if and only if) Let $x, y \in \mathbb{R}$ & $x < y$ then $\exists r \in \mathbb{Q}$ s.t. $x < r < y$

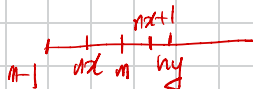
(Note) 3rd case $0 \leq x < y$ or $x < 0$ $\left\{ \begin{array}{l} x < 0 < y \rightarrow r = 0 \\ x < y \leq 0 \Leftrightarrow 0 \leq -y < -x \end{array} \right.$

thus, enough to consider $0 \leq x < y$.

pf By Archimedean Property, $\exists n \in \mathbb{N}$ s.t. $n(y-x) > 1$
 $\Leftrightarrow \underline{n(x+1) < ny}$

By well-ordered principle, $\exists$ smallest m s.t. $nx < m$

$$m-1 \leq nx < m \quad m \leq n(x+1), \quad nx < m < ny \quad \text{next line}$$



$$x < \frac{m}{n} < y \quad \square$$

Ex 1.2.6. $\forall n \geq 2$
 Thm 1.5.3. Let $n \in \mathbb{N}$, for $\forall x > 0 \Rightarrow y > 0$ s.t. $y^n = x$ xy는 제곱차.

remind $a, b > 0$. $a > b \Leftrightarrow a^n > b^n$

pt) Let $E = \{t \in \mathbb{R} : t > 0 \ \& \ t^n < x\}$

claim $y = \sup E$. ← i), ii)에 의해 증명.

i) $\left(\frac{x}{x+1}\right)^n = \frac{x^n}{(x+1)^n} < \frac{x^n}{x^n} = 1 < x$ $\forall x \geq 1$ $E \neq \emptyset$

ii) $t^n < x < (1+x)^n$ $\forall x \geq 1$ $1+x$ is an upper bound of E

we need to show $y^n = x$

Suppose $y^n \neq x$

A) $y^n < x$

B) $y^n > x$

$\hookrightarrow y+h \in E$

for some $h > 0$.

$\hookrightarrow (y+k)^n > x$ for

some $k > 0$

→ 상한선에 닿음.

→ 하한선에 닿음.

값을 찾아볼 것.

A) $(y+h)^n - y^n = h(y^{n-1} + \dots + y^{n-1}) < h \cdot n \cdot y^{n-1} < x - y^n$

take $0 < h < \min\left(\frac{x-y^n}{n(y^{n-1}+1)}, 1\right)$

$\Rightarrow (y+h)^n < x$ $\forall x$

B) $y^n - (y-k)^n < k \cdot n \cdot y^{n-1} < y^n - x$

take $0 < k < \min\left(\frac{y^n-x}{n \cdot y^{n-1}}, y\right)$

$\Rightarrow (y-k)^n > x$ $\forall x$

§1.6. Binary and Ternary Expansions.

Def. (Ternary Expansion) Let $x \in [0, 1]$. Define $\{n_k\}_{k=1}^{\infty}$ as follows:

1) Let $n_k \in \{0, 1, 2\}$

2) n_1 be the largest number that satisfies $\frac{n_1}{3} \leq x$

3) suppose $n_1, \dots, n_k$ are chosen. Then let n_{k+1} be the largest s.t. $\sum_{j=1}^k n_j 3^{-j} + n_{k+1} 3^{-(k+1)} \leq x \left(\leq \left(\sum_{j=1}^k n_j 3^{-j} \right) + 3^{-(k+1)} \right)$.

Thm 1.6.3. Let $E = \left\{ \sum_{j=1}^k n_j 3^{-j} : k=1, 2, 3, \dots \right\}$, then $\sup E = x$

pf $E \neq \emptyset$, E is bounded above (< 1).

$\sup E$ exists. & $x = \sup E < \overset{\text{u.b.}}{1}$.

$$\sup E - \varepsilon \in E.$$

$$\therefore 0 \leq x - \left(\sum_{j=1}^k n_j 3^{-j} \right) \leq 3^{-(k+1)}.$$

$k \rightarrow \infty$ $k \rightarrow \infty$

§1.7. Countable & Uncountable Sets.

Def Let A, B be sets.

We say, A & B have the same cardinality (i.e. A, B are cardinally equivalent)

if there is an 1-1 onto function $f: A \rightarrow B$.

We write $\sim$ by $A \sim B$.

$\Rightarrow$ i) $A \sim A$ ii) if $A \sim B$, $B \sim A$. iii) if $A \sim B$ & $B \sim C$, $A \sim C$

Def Let $N_n = \{1, 2, \dots, n\}$

- i) A is finite if $A \sim N_n$ for some n or $A = \emptyset$
- ii) A is infinite if A is not finite.
- iii) A is countable if $A \sim \mathbb{N}$
- iv) A is uncountable if A is neither finite nor countable.
- v) A is at most countable if A is finite or countable

Ex (a) $\mathbb{N}$ countable

(b) $\mathbb{Z}$ countable. ($\mathbb{N} \sim \mathbb{Z}$)

(c) Every infinite subset of $\mathbb{N}$ is countable

pf) Let $E \subset \mathbb{N}$ be an infinite subset

Let $\tau_1 \in E$ be the smallest element of E

$\tau_2 \in E$ be the smallest element of $E \setminus \{\tau_1\}$.

Continuing the process above, we have

$$\tau_1 < \tau_2 < \tau_3 \dots, \quad E = \{\tau_1, \tau_2, \dots\}$$

$$f(n) = \tau_n, \quad f: \mathbb{N} \rightarrow E \text{ is 1-1 \& onto}$$

(d) $\mathbb{N} \times \mathbb{N}$ is countable

pf) Let $(m, n) \in \mathbb{N} \times \mathbb{N}$. ($\rightarrow \mathbb{N}$).

$$i) f(m, n) = 2^m(2n-1) \rightarrow f \text{ is onto.}$$

$$ii) f(m_1, n_1) = f(m_2, n_2) \Leftrightarrow 2^{m_1}(2n_1-1) = 2^{m_2}(2n_2-1),$$

$$n_1 = n_2, \quad m_1 = m_2 \rightarrow f \text{ is 1-1}$$

$$\Rightarrow \mathbb{N} \times \mathbb{N} \times \mathbb{N} \times \dots \subseteq \mathbb{H}_0.$$

$$g(m, n, l) = 2^{f(m, n)} (2l-1) \dots$$

Def (Sequences) Let A be a set, A sequence in A is a fun $f: \mathbb{N} \rightarrow A$.

• for $n \in \mathbb{N}$, $\underbrace{d_n = f(n)}$ (nth term).

Def (Enumeration of Countable Set A). Let A be a countable set.

$\exists$ 1-1 & onto $f: \mathbb{N} \rightarrow A$, $d_n = f(n)$. $A = \{d_n \mid n=1, 2, \dots\}$.
 $\hookrightarrow$ enumeration of A

* countable set $\Leftrightarrow$ sequence $\Leftrightarrow$ ~~set~~ $\mathbb{H}_0$.

$\{d_n\}_{n=1}^{\infty}$ is called "enumeration of A ".

Thm 1.1.6. Every infinite subset E of a countable set A is countable.

pf) $\exists$ 1-1 & onto $f: \mathbb{N} \rightarrow A$.

$E \subset A$ or $\mathbb{R} \subseteq \mathbb{Z}$, Some $f^{-1}(E) \subset \mathbb{N}$, and

$f^{-1}(E)$ is infinite, thus countable. (for $E \subset \mathbb{R}$).

$f: f^{-1}(E) \rightarrow E$ is 1-1 & onto

$\mathbb{N} \sim f^{-1}(E) \sim E$, E is countable $\square$

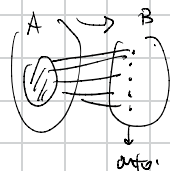
Thm 1.1.1. Let A be a countable set.

i) B is (at most) countable if there is an onto $f: A \rightarrow B$.

Ex $f: \mathbb{Z} \times \mathbb{Z} \rightarrow \mathbb{Q}$ (* $\mathbb{Z} \times \mathbb{Z} \subseteq$ countable
 since A, B countable $\Rightarrow A \times B$ countable).
 $f(m, n) = \begin{cases} \frac{m}{n} & \text{if } n \neq 0 \\ 0 & \text{if } n = 0 \end{cases} \Rightarrow f$ is onto, $\mathbb{Q}$ is at most countable.

pt) Let $b \in B$, $f^{-1}(f(b_1)) \cap f^{-1}(f(b_2)) = \emptyset$ if $b_1 \neq b_2$

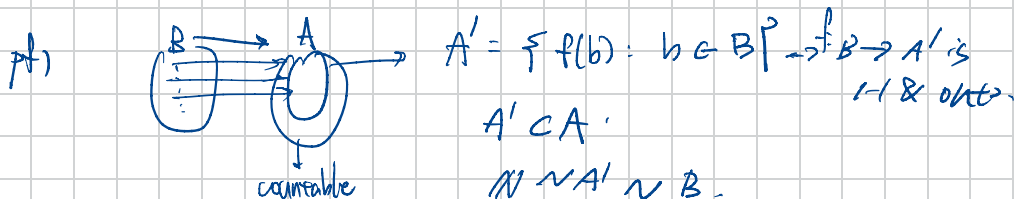
Let $a(b) \in f^{-1}(f(b))$, $b \mapsto a(b)$ is 1-1.



$$A' = \{a(b) : b \in B\} \subset A.$$

$\hookrightarrow$ A' is at most countable. $B \rightarrow A'$ is 1-1 onto.

2) B is (at most) countable if there is an 1-1 $f: B \rightarrow A$.



Def (Indexed family of sets) Let $A \times X$ be a set. An indexed family of set with index A is a fun $A \rightarrow \mathcal{P}(X)$. Let $f: A \rightarrow \mathcal{P}(X)$ for each $\alpha \in A$. $E_\alpha = f(\alpha)$... $\{E_\alpha : \alpha \in A\}$ is a set.

Def $\bigcap_{\alpha \in A} E_\alpha = \{x \in X : x \in E_\alpha \text{ for all } \alpha \in A\}$

$\bigcup_{\alpha \in A} E_\alpha = \{x \in X : x \in E_\alpha \text{ for some } \alpha \in A\}$

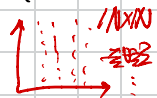
Def $\mathcal{E} \subset \mathcal{P}(X)$

$$\bullet E \cap \left(\bigcup_{\alpha \in A} E_\alpha \right) = \bigcup_{\alpha \in A} (E \cap E_\alpha) \quad \bullet \left(\bigcup_{\alpha \in A} E_\alpha \right)^c = \bigcap_{\alpha \in A} E_\alpha^c$$

$$\bullet E \cup \left(\bigcap_{\alpha \in A} E_\alpha \right) = \bigcap_{\alpha \in A} (E \cup E_\alpha) \quad \bullet \left(\bigcap_{\alpha \in A} E_\alpha \right)^c = \bigcup_{\alpha \in A} E_\alpha^c$$

Thm 1.1.15. Let $\{E_n\}_{n \in \mathbb{N}}$ is a sequence of countable sets.

then $\bigcup_{n=1}^{\infty} E_n$ is countable.



pf) Since E_n is countable, $\exists$ an enumeration $\{d_{n,k}\}_{k=1}^{\infty}$ of E_n
for $(n,k) \in \mathbb{N} \times \mathbb{N}$ s.t. $f(n,k) = d_{n,k} \rightarrow f: \mathbb{N} \times \mathbb{N} \rightarrow \bigcup_{n=1}^{\infty} E_n$
 f is onto. by thm 1.1.1, $\bigcup_{n=1}^{\infty} E_n$ is (at most) countable

Cor. $\mathbb{Q}$ is countable

pf) $\mathbb{Q} = \bigcup_{n=1}^{\infty} \left\{ \frac{k}{n} : k \in \mathbb{Z} \right\}$

$E_n = \left\{ \frac{k}{n} : k \in \mathbb{Z} \right\}$ is countable $\xrightarrow{\text{thm 1.1.15}} \mathbb{Q}$ is countable

Uncountable? $\Rightarrow$ countable set of $\mathbb{R}$ is uncountable.

(A is uncountable, B is countable subset of $A \Rightarrow A \setminus B$ uncountable)

Thm $U = \left\{ (d_1, d_2, \dots) : d_k \in \{0,1\}, k=1,2,\dots \right\}$ is uncountable.
 $\hookrightarrow \forall f: \mathbb{N} \rightarrow \{0,1\}$.

$x_k = (d_1, d_2, \dots)$

pf) Suppose not: U is countable $\Rightarrow \exists$ an enumeration $\{x_k\}_{k=1}^{\infty}$ of U .
by contradiction

Define $Y = (y_1, y_2, \dots)$ $y_k = \begin{cases} 1 & \text{if } d_{kk} = 0 \\ 0 & \text{if } d_{kk} = 1 \end{cases}$

then $Y \neq x_k$ for all k , but $Y \in U$.

Cor the interval $[0,1]$ is uncountable.

pf) $x \in [0,1]$ $\exists$ binary expansion: $x = 0.d_1d_2d_3\dots$ ($d_k \in \{0,1\}$)
is unique.

$\rightarrow [0,1] = U$.

Ch 2. Topology of the Real Line

§2.1. Metric Spaces (Metric)

Def (Metric space (X, d)) Let X be a nonempty set.

We say X is a metric space with a metric $d: X \times X \rightarrow \mathbb{R}$
(distance fun)

satisfying:

1) $d(x, y) \geq 0 \quad \forall x, y \in X$

2) $d(x, y) = 0 \iff x = y$

3) $d(x, y) = d(y, x)$

4)* (Triangle inequality) $d(x, y) \leq d(x, z) + d(z, y) \quad \forall x, y, z \in X$

Ex A • $X = \mathbb{R} \quad d(x, y) = |x - y| \quad x, y \in \mathbb{R}$

• $X = \mathbb{R}^d$
$$\begin{cases} d(x, y) = \sqrt{(x_1 - y_1)^2 + \dots + (x_d - y_d)^2} \\ d(x, y) = \max_{1 \leq i \leq d} |x_i - y_i| \end{cases} \quad \begin{matrix} x: (x_1, \dots, x_d) \in \mathbb{R}^d \\ y: (y_1, \dots, y_d) \in \mathbb{R}^d \end{matrix}$$

B $\mathbb{Z}$ 의 set, $d(x, y) = \begin{cases} 1 & x \neq y \\ 0 & x = y \end{cases} \Rightarrow$ 2-nd of point y
 $\rightarrow \forall x, d(x, y) < 2$
 $=$ 전체 $\mathbb{Z}$ 의 X .

C Let $A \subset \mathbb{R}$ &

$$X = \{f: f: A \rightarrow \mathbb{R} \mid \sup_{x \in A} |f(x)| < \infty \text{ for some } M\}$$

$$d(f, g) = \sup_{x \in A} |f(x) - g(x)|$$

$\hookrightarrow$ 1), 3) $\Leftarrow$ trivial, 2) $\Leftarrow d(f, g) = 0 \iff f(x) = g(x) \text{ for } \forall x \in A$.

*) Let $x \in A$ & $h \in X$, $|f(x) - g(x)| \leq |f(x) - h(x)| + |h(x) - g(x)|$

$$\begin{aligned} d(f, g) &= \sup_{x \in A} |f(x) - g(x)| \\ &\leq \sup_{x \in A} |f(x) - h(x)| + \sup_{x \in A} |h(x) - g(x)| \\ &= d(f, h) + d(h, g) \end{aligned}$$

$\hookrightarrow$ upper bound.

C' $C(A) = \{f: f: A \rightarrow \mathbb{R}, \text{continuous \&}$
 $\sup_{x \in A} |f(x)| < M \text{ for some } M\}$

ex. $A = I = [a, b] : \exists \epsilon m - \exists \epsilon \neq \exists \epsilon$

§ 2.2. Open & Closed Sets

Def (ϵ -neighborhood of a point p) Let (X, d) be a metric space.

An ϵ -nbd of a point $p \in X$ is the set $\{x \in X, d(x, p) < \epsilon\}$

We denote it by $N_\epsilon(p)$

$\hookrightarrow \epsilon = 0$ 이면 1 point
 $\epsilon < 0$ 이면 $\emptyset$.
 (모든 nbd)

Ex $X = \mathbb{R}, d(x, y) = |x - y| \Rightarrow N_\epsilon(x) = (x - \epsilon, x + \epsilon)$

Def (Interior point) Let (X, d) be a metric space & $E \subset X$.

We say 'a point $p \in X$ is an interior point of E ' if $\exists \epsilon > 0$,
 s.t. $N_\epsilon(p) \subset E$

• Set $\text{Int}(E) = \{x \in X : x \text{ is an interior point of } E\} \Rightarrow \text{Int}(E) \subset E$

Ex A) $X = \mathbb{R}, d(x, y) = |x - y|, E = (a, b)$.

• if $x \in (a, b), \epsilon = \min(|x - a|, |x - b|) / n \geq \text{arbitrary}$

$N_\epsilon(x) \subset (a, b) \Rightarrow x \in \text{Int}(E)$

• if $x = a, b$, or $x < a, x > b$, then x is not an interior point.

$\therefore \text{Int}(a, b) = (a, b)$

B) $X = \mathbb{R}, E = [a, b]. \Rightarrow \text{Int } E = (a, b)$.

C) $X = \mathbb{R}, E = \mathbb{R} \setminus \mathbb{Q} \Rightarrow$ Let $x \in \mathbb{R}$, there is no $\epsilon > 0$
 s.t. $N_\epsilon(x) \subset E$.

$\therefore E$ has no interior point ($E = \mathbb{Q}$ is also not interior).

Def Let (X, d) be a metric space & $E \subset X$

1) We say " E is open" if $\text{Int } E = E$

($\Leftrightarrow$) for $\forall p \in E \quad \exists \varepsilon > 0$ s.t. $N_\varepsilon(p) \subset E$

2) We say " E is closed" if E^c is open.

Ex $X = \mathbb{R}$, (a, b) , $(-\infty, a)$, (b, ∞) , $[a]^c = (-\infty, a) \cup (a, \infty)$ are open.

$\bullet$ $[a_1, \dots, a_n]$ ($a_1 < \dots < a_n$), $[a, b]$ are closed.
 * union of open sets = open.

$\bullet$ $\mathbb{Q}$ is not open, neither is $\mathbb{R} \setminus \mathbb{Q} \Rightarrow \mathbb{Q}$ is neither open nor closed.

Thm 2.2.8. Every ε -nbd of X ($\varepsilon > 0$) is open.

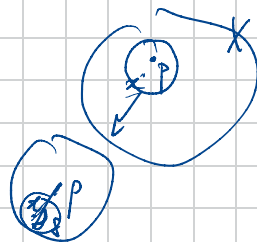
pf) Let $\varepsilon > 0$ & $p \in X$, consider $N_\varepsilon(p)$

Let $x \in N_\varepsilon(p)$ and $\delta = \varepsilon - d(x, p)$

Let $y \in N_\delta(x)$. $d(x, y) < \delta$.

Claim: $d(y, p) < \varepsilon$. ($\Leftrightarrow N_\delta(x) \subset N_\varepsilon(p)$)

$d(y, p) \leq d(y, x) + d(x, p) < \delta + d(x, p) = \varepsilon$.



Thm 2.2.9. Let (X, d) be a metric space, then.

1) Suppose $\{O_\alpha\}_{\alpha \in A}$ is a family of open subsets of X . Then $\bigcup_{\alpha \in A} O_\alpha$ is open.

pf) Let $x \in \bigcup_{\alpha \in A} O_\alpha$. Then $x \in O_\alpha$ for some α . Since O_α is open,

$\exists \varepsilon > 0$ s.t. $N_\varepsilon(x) \subset O_\alpha$ then $N_\varepsilon(x) \subset \bigcup_{\alpha \in A} O_\alpha$

2) Suppose $O_1, \dots, O_n$ be open subsets of X . Then, $\bigcap_{k=1}^n O_k$ is open.

pf) Let $x \in \bigcap_{k=1}^n O_k$. $x \in O_k$ for all $1 \leq k \leq n$

Since O_k is open, $\exists \epsilon_k > 0$ s.t. $N_{\epsilon_k}(x) \subset O_k$.

Set $\epsilon = \min\{\epsilon_1, \dots, \epsilon_n\}$. then $N_\epsilon(x) \subset O_k \subset \bigcap_{k=1}^n O_k$ (for $\forall k$)

1) If $\{F_\alpha\}_{\alpha \in A}$ & F_α are closed, $\bigcap_{\alpha \in A} F_\alpha$ is closed.

pf) $(\bigcap_{\alpha \in A} F_\alpha)^c = \bigcup_{\alpha \in A} F_\alpha^c$ is open.

2) If $F_1, \dots, F_n$ are closed, $\bigcup_{k=1}^n F_k$ is closed.

pf) $(\bigcup_{k=1}^n F_k)^c = \bigcap_{k=1}^n F_k^c$ is open

Ex a) $O_n = (-\frac{1}{n}, \frac{1}{n}) \Rightarrow \bigcap_{n=1}^{\infty} O_n = \{0\}$, closed.

b) $\{X_n\}_{n \in \mathbb{N}}$ be an enumeration of $\mathbb{Q} \Rightarrow \bigcap_{n=1}^{\infty} (\mathbb{R} \setminus \{X_n\}) = \mathbb{R} \setminus \mathbb{Q}$

Def Let X be a metric space & $E \subset X$.

1) (^{= limit pt.} Limit point) We say a point $p \in X$ is a limit point of E if
"every ϵ -nbd of p contains (at least) an element of E ($\neq p$)"
($\epsilon > 0$)

($\Leftrightarrow \forall \epsilon > 0 \quad (N_\epsilon(p) \setminus \{p\}) \cap E \neq \emptyset$)

2) We say a point p is an isolation point of E if
 $p \in E$ is not a limit point of E .

Ex a) $X = \mathbb{R}$, $E = (a, b]$

$\begin{cases} \text{If } x \in (a, b), x \text{ is limit point of } E. \\ \text{If } x = a \text{ or } b, \end{cases}$ "

If $x < a$ or $x > b$, x is not a limit point.

b) $E = \{\frac{1}{n}; n=1, 2, \dots\}$

$\begin{cases} \text{If } x \neq 0, x \text{ is not a limit point of } E. \\ \text{If } x = 0, x \text{ is a limit point of } E. \end{cases}$

$\Rightarrow E \setminus \{0\}$ points are isolated points

Thm 2.2.15. Let X be a metric space & $E \subset X$.

If p is a limit point of E , then every ε -nbd of p contains infinitely many elements of E .

Cor. If E is a finite set, then E has no limit points.

pf) Suppose there is an $\varepsilon_0 > 0$ such that $N_{\varepsilon_0}(p) \cap E$ is a finite set.

i.e., $(N_{\varepsilon_0}(p) \setminus \{p\}) \cap E = \{q_1, \dots, q_k\} (\neq \emptyset)$.

Let $\delta = \min \{d(p, q_i) > 0\}$, $(N_\delta(p) \setminus \{p\}) \cap E = \emptyset$.

p is not a limit point $\neq \square$

Thm 2.2.14 Let $F \subset X$ then F is closed iff F contains every limit point of itself.

pf) $(\Rightarrow)$ Let p be a limit point of $F \Rightarrow p \in F$ or not
 $p \in F^c$

Suppose $p \in F^c$. Since F^c is open,

$\exists$ an ε -nbd $N_\varepsilon(p) \subset F^c$, $N_\varepsilon(p) \cap F = \emptyset \Rightarrow p \notin F$ or not.

p is not a limit point of $F \therefore p \in F$ or not.

$p \notin F$ or not
not a limit point
(for ε small enough)

(*) Need to show F^c is open.

Let $p \in F^c$, p is not a limit point of F .

For some $\varepsilon > 0$, $(N_\varepsilon(p) \setminus \{p\}) \cap F = \emptyset$

$$N_\varepsilon(p) \cap F = \emptyset \Rightarrow N_\varepsilon(p) \subset F^c$$

$\therefore F^c$ is open.

Def Let $E \subset X$, E' derived set

Define $E' = \{x \in X : x \text{ is a limit point of } E\}$

$$\bar{E} = E \cup E'$$

$\hookrightarrow$ closure

$\rightarrow$ closed? E of limit point + $\bar{E}$ of limit point
 $\rightarrow$ $\bar{E}' = \bar{E}' = E' (*)$
 $\rightarrow$ $\bar{E}$ is closed.

pf of *) (i) $\bar{E}$ is closed since if $F \subset E$, $F \subset E'$.

(c) Let $p \in (\bar{E})'$

* notation. $N_\varepsilon^\circ(p) = N_\varepsilon(p) \setminus \{p\}$.

claim: $\forall \varepsilon > 0$, $N_\varepsilon^\circ(p) \cap E \neq \emptyset$ ($\Leftrightarrow p$ is limit point of E)

Given $\varepsilon > 0$, since $p \in (\bar{E})'$, $N_{\varepsilon/2}^\circ(p) \cap (\bar{E}) \neq \emptyset$

$\exists q \in N_{\varepsilon/2}^\circ(p) \cap (\bar{E})$ $\Rightarrow$ if $q \in E$, $N_{\varepsilon/2}^\circ(q) \cap E \neq \emptyset$

if $q \in E'$, $N_{\varepsilon/2}^\circ(q) \cap E \neq \emptyset$ (argued def.)

$\exists r \neq q \in N_{\varepsilon/2}^\circ(p) \cap E$. (by thm 2.2.1b)

$\therefore r \in N_\varepsilon^\circ(p) \cap E \Rightarrow N_\varepsilon^\circ(p) \cap E \neq \emptyset$

* isolated - limit point is exclusive.

$\exists$ interior points (limit point) $\exists$ $\bar{E}$ of X . / interior points
 $\exists \varepsilon$ -nbhd $N_\varepsilon(p) \subset E$. $\forall N_\varepsilon^\circ(p)$ contains dense of E .
 isolated point $\notin \bar{E}$.

Thm 2.2.4. Let $E \subset X$, then $\overline{(E \cup E')} = \overline{E} = (E \cup E')'$ ^{should be} _{or is it?}

1) $\overline{E}$ is closed.

pf) $(E \cup E')' = E' \cup E''$ $E' \subset (E \cup E')' = \overline{E}$ $\square$

2) $E = \overline{E}$ iff E is closed.

pf) (a) if $E = \overline{E}$, E is closed. by 1)

($\Leftarrow$) if E is closed, $E' \subset E$ therefore $E = E \cup E' = \overline{E}$ $\square$

3) if $E \subset F$ & F is closed, then $\overline{E} \subset F$.

($\Leftarrow$) $\overline{E}$ is the smallest closed set containing E .

pf) Since F is closed, $F' \subset F$ and since $E \subset F$, $E' \subset F'$

$E' \subset F$, thus $(E \cup E')' = \overline{E} \subset F$ $\square$

Def Let X be a metric space. Let $D \subset X$.

D is dense in X if $\overline{D} = X$.

$\overline{D} \supset X \rightarrow$ not false $\overline{D} \subset X \rightarrow$ by thm 2.2.8. (a) X $\subset$ D' ^{is not always..} on D

($\Leftarrow$) if for any $x \in X$ & $\epsilon > 0$, $\exists y \in D$ s.t. $y \in N_\epsilon(x)$

Ex. $\mathbb{Q} \subset \mathbb{R}$

pf) let $x \in \mathbb{R}$ & $\epsilon > 0$.

by thm 1.5.2. $\exists q \in \mathbb{Q}$ s.t. $q \in (x - \epsilon, x + \epsilon)$.

thus, every point of $\mathbb{R}$ is a limit point of $\mathbb{Q}$. $\mathbb{R} \subset \mathbb{Q}'$ $\square$

$\mathbb{R} \setminus \mathbb{Q}$ is also dense in $\mathbb{R}$.

Thm 2.2.20. Let $O \subset \mathbb{R}$ open. Then there exists mutually disjoint open intervals $\{O_n\}$, at most countable s.t. $O = \bigcup_n O_n$ ↳ if $O_n \cap O_m \neq \emptyset$, $O_n = O_m$

Pr) Let $x \in O$. $\exists \varepsilon > 0$ s.t. $(x - \varepsilon, x + \varepsilon) \subset O$.

Let $l_x = \inf \{y : (y, x) \subset O\}$ and $r_x = \sup \{y : (x, y) \subset O\}$ and $O_x = (l_x, r_x)$

then 1) $O_x \subset O$, 2) If $x_1, x_2 \in O$, either $O_{x_1} = O_{x_2}$ or $O_{x_1} \cap O_{x_2} = \emptyset$

For each O_x , choose $q_x \in O_x$ s.t. $x \in O_x$. then if O_{x_1} & O_{x_2} are distinct,

then $q_{x_1} \neq q_{x_2}$. thus $f: \{O_x\} \rightarrow \mathbb{Q}$ s.t. $O_x \rightarrow q_x$ is 1-1,

therefore $\{O_x\}$ is at most countable.

$\bigcup_n O_n = O$ and by 1) $\{O_x\}$ is mutually disjoint.

Def (Connected, Disjoint Set) A is disjoint if there exist disjoint open set O_1, O_2 such that $A \cap O_1 \neq \emptyset$ and $A \cap O_2 \neq \emptyset$.

We say A is connected if A is not disjoint.

Def (Relatively Open & Closed Set) Let $Y \subset X$, $E \subset Y$, $F \subset Y$.

1) We say E is open relative to Y if for $\forall p \in E$,

$\exists \varepsilon > 0$ s.t. $N_\varepsilon(p) \cap Y \subset E$.

Ex $\mathbb{Q} \subset \mathbb{R} \Rightarrow$ not open nor closed / $\mathbb{Q}$ is open relative to $\mathbb{Q}$

2) We say F is closed relative to Y if F^c is open relative to Y .

Thm. Let $Y \subset X$, $E \subset Y$.

1) E is open relative to F iff there is an open set

$$O \subset X \text{ s.t. } E = Y \cap O$$

A) $\Leftrightarrow$ trivial. Since O is open $\exists \varepsilon > 0$ s.t. $N_\varepsilon(y) \subset O$.

$$\left(\begin{array}{c} \text{E} \\ \text{E} \\ \text{E} \end{array} \right)$$

$$A \Rightarrow \text{t. } \underbrace{N_\varepsilon(\varphi)} \cap Y \subset \underbrace{\partial Y}$$

$$\Leftrightarrow \text{for each } p \in E, \exists \varepsilon_p > 0 \text{ s.t. } N_{\varepsilon_p}(p) \cap Y \subset E.$$

Set $\mathcal{O} = \bigcup_{p \in E} N_{\text{eq}}(p)$. Prove that $\mathcal{O} \cap Y = E$.

$$\left(\bigcup_{p \in E} N_{\text{ap}}(p) \right) \cap Y$$

$$= \bigcup_{p \in E} (N(p) \cap Y)$$

남 def. 2E 앞은 개명.

$$= \bigcup_{p \in E} (N_{eq}(p) \cap Y)$$

2) E is closed relative to Y iff there is a closed set

$$FCX \text{ s.t. } E = Y \cap F$$

§ 2.3. Compact Sets.

Def Let $E \subset X$, A collection $\{O_\alpha\}_{\alpha \in A}$ of open subset of X is an open cover of E if $\bigcup_{\alpha \in A} O_\alpha \supset E$.

Ex $\mathbb{R} = \bigcup_{n=1}^{\infty} (-n, n) = \bigcup_{n=-\infty}^{\infty} (n-1, n+1)$

Def Let $K \subset X$, We say K is compact if every open cover K has a finite subcover

$\Leftrightarrow$ if O_α is open and $\bigcup_{\alpha \in A} O_\alpha \supset K$, $\exists O_{\alpha_1} \dots O_{\alpha_n}$ s.t. $\bigcup_{j=1}^n O_{\alpha_j} \supset K$.
($\alpha_1 \dots \alpha_n \in A$)

Counter Ex 1. $\mathbb{R} = \bigcup_{n=1}^{\infty} (-n, n) \Rightarrow$ ~~finite subcover of $\mathbb{R}$~~ not opt.

Ex 2 $[-a, b] \subset \bigcup_{n=1}^{\infty} (-a + \frac{1}{n}, b+1) \Rightarrow$ not opt.

Ex 3 $K = \{p\} \Rightarrow$ suppose $\bigcup_{\alpha \in A} O_\alpha \supset K$ for a collection $\{O_\alpha\}_{\alpha \in A}$ of open set.
 $\exists$ an $O_{\alpha_0} \ni p \rightarrow$ ~~finite subcover of K~~

+ A finite set is opt

Thm ^{2.3.4(a)} Let $K \subset X$ be a opt set. Then K is closed.

pf We show K^c is open

Let $p \in K^c$. for each $q \in K$ $d(p, q) > 0$.

Consider $\bigcup_{q \in K} N_{\frac{d(p, q)}{2}}(q)$, not.

Since K is compact, $\exists N_{\frac{d(p, q_i)}{2}}(q_i)$, $i=1, \dots, l$ s.t. $\bigcup_{i=1}^l N_{\frac{d(p, q_i)}{2}}(q_i) \supset K$.

Set $\delta = \min_{i=1, \dots, l} (\frac{d(p, q_i)}{2})$, $N_\delta(p) \cap K = \emptyset$

($\because N_\delta(p) \cap \bigcup_{i=1}^l N_{\frac{d(p, q_i)}{2}}(q_i) = \emptyset$)

$\Rightarrow N_\delta(p) \subset K^c$ $\square$



2.27.(b)

Thm Let $K \subset X$, Suppose K is cpt & $F \subset K$ is closed.

Then F is cpt.

(cpt set & closed subset $\subseteq$ cpt)

pf) Let $\{O_\alpha\}_{\alpha \in A}$ be an open cover of F .

Since F is closed, F^c is open.

$\{O_\alpha\}_{\alpha \in A} \cup F^c$ is an open cover of K .

$$(\Leftrightarrow \bigcup_{\alpha \in A} O_\alpha \cup F^c \supset K)$$

$\bigcup_F$

$\rightarrow$ ~~finite~~

Since K is cpt, $\exists$ finite subcover $O_{\alpha_1}, \dots, O_{\alpha_n}, F^c$ s.t. $\bigcup_{j=1}^n O_{\alpha_j} \cup F^c \supset K$.

Note $F \subset K$, so $F \subset \bigcup_{j=1}^n O_{\alpha_j}$ $\square$

2.3.6.

Cor Let $K, F \subset X$,

Let F be closed & K is cpt, then $F \cap K$ is cpt.

2.3.7.

Thm Let E be an infinite subset of a cpt set K ,

Then E has a limit point in K

(1) $\underline{\hspace{1cm}}$

(2) $\underline{\hspace{1cm}}$

pf) Since K is closed,

if a limit point p of E exists, then $p \in K$. (2)

Suppose there is no limit point of E . ($\rightarrow E$ is closed ($E = \bar{E}$)
 $\rightarrow E$ is cpt

For $p \in K$, $\exists \varepsilon_p > 0$ such that

$N_{\varepsilon_p}(p)$ contains at most one element of E . ($p \in E$ or not, at least one)

$$\bigcup_{p \in K} N_{\varepsilon_p}(p) \supset K.$$

Since K is cpt, $\exists$ finite subcover $N_{\varepsilon_{p_1}}(p_1), \dots, N_{\varepsilon_{p_n}}(p_n)$ of K .

In particular, $\bigcup_{j=1}^n N_{\varepsilon_{p_j}}(p_j) \supset E \Rightarrow E$ is a finite set. $\square$

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Thm (Nested Set Property) Let $K_n \subset X$ be nonempty cpt sets.

Suppose $K_n \supset K_{n+1} \quad \forall n \geq 1$, then $\bigcap_{n=1}^{\infty} K_n \neq \emptyset$.

Pr) Suppose $\bigcap_{n=1}^{\infty} K_n \neq \emptyset$, then $\bigcup_{n=1}^{\infty} K_n^c = X$ & K_n^c are open.

$$\bigcup_{n=1}^{\infty} K_n^c \supset K_1$$

Since K_1 is cpt, $\exists$ a finite subcover $K_{n_1}^c, \dots, K_{n_k}^c$ of K_1
 $(n_1 < \dots < n_k)$

$$\bigcup_{j=1}^k K_{n_j}^c \supset K_1, \quad \rightarrow$$

$$\bigcup_{j=1}^k K_{n_j} = K_{n_j} \subset K_1^c \quad \& \quad K_{n_j} \subset K_1, \quad K_{n_j} = \emptyset \quad \text{or}$$

$\oplus \bigcap_{n=1}^{\infty} K_n$ is closed & cpt.

§ 2.4. Compact Subsets of $\mathbb{R}^k$

Def Let $a_j \leq b_j, 1 \leq j \leq k$.

$Q = [a_1, b_1] \times \dots \times [a_k, b_k] \subset \mathbb{R}^k$ is called a **k-cell**.

Def A set $E \subset \mathbb{R}^k$ is bounded if $\exists M > 0$ s.t. $|x| \leq M \quad \forall x \in E$.

E is unbounded if for all $M > 0 \exists x \in E$ s.t. $|x| > M$

Lemma. Let $\{I_n\}_{n=1}^{\infty}$ be a sequence of closed intervals, opt by this ~~not~~ X.

Suppose $I_n \supset I_{n+1} \quad \forall n \geq 1$, then $\bigcap_{n=1}^{\infty} I_n \neq \emptyset$.

Pr) Let $I_n = [a_n, b_n]$. Since $I_n \supset I_{n+m} = [a_{n+m}, b_{n+m}]$ & $I_n \supset I_{n+1}$

$$a_n \leq a_{n+m} \leq b_{n+m} \leq b_n \quad \text{for } \forall n, m \geq 1$$

$\alpha = \sup_n a_n$ ($\leftarrow$ Least Upper Bound Property: a_n is upper bounded & nonempty)

$$a_n \leq \alpha \leq b_n, \quad \forall n \geq 1, \quad \alpha \in \bigcap_{n=1}^{\infty} [a_n, b_n], \quad \therefore \bigcap_{n=1}^{\infty} I_n \neq \emptyset \quad \text{III}$$

Lemma Let $\{Q_n\}_{n=1}^{\infty}$ be a sequence of k -cells. Suppose $Q_n \supset Q_{n+1}$, $\forall n \geq 1$.
 then $\bigcap_{n=1}^{\infty} Q_n \neq \emptyset$.

pf) $Q_n = I_n^1 \times \dots \times I_n^k$. then $I_n^j \supset I_{n+1}^j \quad \forall n \geq 1$ for $j=1 \dots k$.
 $I_{\infty}^j := \bigcap_{n=1}^{\infty} I_n^j \neq \emptyset$, $I_{\infty}^1 \times \dots \times I_{\infty}^k \subset \bigcap_{n=1}^{\infty} Q_n$
 $\hookrightarrow \neq \emptyset$ ↑ 2个 1个! □

Thm Every k -cell is cpt.

pf) Enough to show

prop. Let $R > 0$ then $Q_R = [-R, R] \times \dots \times [-R, R]$ is cpt.

< $\because$ Let Q be a k -cell, then $\exists R$ st. $Q \subset Q_R$.

since Q_R is cpt & Q is closed, Q is cpt)

Suppose Q_k is not cpt.

$\exists$ an open cover $\{O_\alpha\}_{\alpha \in A}$ of Q_k that has no finite subcover.

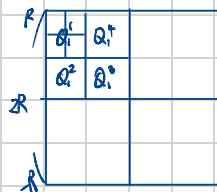
Divide Q_k into 4 equal closed squares. $Q_1^1, Q_1^2, Q_1^3, Q_1^4$.

$\exists Q_1 \subset Q_k$ that has no finite subcover. (~~finite subcover~~ stayed ~~st~~)

$\exists Q_2 \subset Q_1$ that has no finite subcover

--- continue the procedure.

- $\Rightarrow$
- 1) $Q_1 \supset Q_2 \supset Q_3 \supset \dots$
 - 2) Q_j has no finite subcover
 - 3) side length of $Q_j = R \cdot 2^{-j}$



By lemma, $\bigcap_{n=1}^{\infty} Q_n \neq \emptyset$ & $\bigcap_{n=1}^{\infty} Q_n = \{p\}$ for some $p \in \mathbb{R}^k$

Then $p \in O_\alpha$ for some α .

Since O_α is open, $\exists \varepsilon > 0$ st. $N_\varepsilon(p) \subset O_\alpha$. if $\forall R \cdot 2^{-n} < \varepsilon$.

then $Q_n \subset N_\varepsilon(p)$, $Q_n \subset O_\alpha$.

$\neq$ □

Cor. Let $E \subset \mathbb{R}^k$. If E is closed & bounded, E is cpt.

pf) $\exists$ an $M > 0$ such that $|x| \leq M, \forall x \in E$.

$E \subset [-M, M] \times \dots \times [-M, M] = Q_n$. Q_n is cpt. ^{by thm}

Since E is a closed subset of a cpt set, E is cpt. $\square$

Thm Let $E \subset \mathbb{R}^k$. The following are equivalent.

1) E is closed and bounded.

$\hookrightarrow$ Cor. above

2) E is cpt.

3) Every infinite subset of E has a limit point in E .

$\hookrightarrow$ thm 2.3.7.

pf) Only need to show $3) \rightarrow 1)$

(E is bounded) Suppose E is not bounded, then for each $n \in \mathbb{N}$

$\exists x_n \in E$ such that $|x_n| \geq n$.

The infinite set $\{x_n\}$ cannot have a limit point. $\times$

$$1 \leq |x_n| < |x_{n+1}|$$

($\because$ any $x_m \in \mathbb{R}^k$, if $n > k+1$, $|x_n - x_m| \geq 1$)

(E is closed) Let p be a limit point of E . Need to show $p \in E$.

Since p is a limit point of E , $\exists p_n \in E$ s.t.

$\times E$ is closed.

$|p - p_n| < 1/n$ for $n=1, 2, \dots$

Then p is a limit point of a infinite subset $\{p_n\}$.

Need to show p is the only limit point of $\{p_n\}$.

Let $q \in \mathbb{R}^k$ & $q \neq p$, $\exists N$ s.t. $|p - p_n| < |p - q|/2$ if $n \geq N$.

$\forall \epsilon$ not zero.

$|q - p_n| \geq |q - p| - |p - p_n| > |p - q|/2$ for $n \geq N$.

q cannot be a limit point of $\{p_n\}$.

therefore p is the unique limit point of $\{p_n\}$. $\therefore p \in E$ $\square$



§ 2.4. The Cantor Set.

Def (Cantor Ternary Set) * Removing the middle third set.

$$\text{Set } P = [0, 1] \setminus \bigcup_{n=0}^{\infty} \bigcup_{k=0}^{2^n-1} \left(\frac{2k+1}{3^{n+1}}, \frac{2k+2}{3^{n+1}} \right)$$

$$= \bigcup_{n=0}^{\infty} \left(\bigcup_{k=0}^{2^{n-1}} \left[\frac{3k}{3^{n+1}}, \frac{3k+1}{3^{n+1}} \right] \cup \left[\frac{3k+2}{3^{n+1}}, \frac{3k+3}{3^{n+1}} \right] \right)$$

$$\hookrightarrow \text{set } p_n$$

then P is epc & $P_n \supset P_{n+1}$ for $\forall n \geq 0$.
 = closed & bounded. $\checkmark$

$$P_{\sigma} = \begin{bmatrix} 0 & 1 & 2 & 1 \end{bmatrix}$$

By Thm 2.3 & Thm 2.4, P is closed and nonempty.

$\mathbb{P}^2$ is called the Cantor ternary set.

$$P = \begin{pmatrix} \bigcup_{k=0}^2 \left[\frac{0}{9} \frac{1}{9} \right] \cup \left[\frac{2}{9} \frac{2}{9} \right] \\ \left[\frac{2}{9} \frac{4}{9} \right] \cup \left[\frac{5}{9} \frac{6}{9} \right] \\ \left[\frac{5}{9} \frac{8}{9} \right] \cup \left[\frac{8}{9} \frac{9}{9} \right] \end{pmatrix}$$

Properties 1) All the endpoints of the intervals are contained in P .

that is to say, $\left\{ \frac{z_{t+1}}{z_t}, \frac{z_{t+1}}{z_t}, \frac{z_{t+1}}{z_t}, \frac{z_{t+1}}{z_t} \right\}_{t=0, \dots, n-1} \in \mathcal{P}$ for any n .

2) Every point of P is a limit point of P ($\Leftrightarrow$ no isolated points)

1b) Let $f \in P$ & $\varepsilon > 0$. Choose n s.t. $3^{-(n+1)} < \varepsilon$.

Note $q \in \bigcap_{n=0}^{\infty} P_n$. So $q \in P_n$. } YH

This $q \in I = [\frac{3k}{3^{n+1}}, \frac{3k+1}{3^{n+1}}]$ or $q \in I' = [\frac{3k+2}{3^{n+1}}, \frac{3k+3}{3^{n+1}}]$ for some $k=0, \dots, 3^n-1$

In such case, we have $I \subset N_\varepsilon(g)$, $I' \subset N_\varepsilon(g)$

since the endpoints of I, I' are contained in P .

There is a point of P in $N_{\mathbb{Q}}(g)$ other than g . $\square$

↳ endpoints 들 중 하나...

 ~~$(\check{e}, \check{i})$~~
9

3) P is totally disconnected, that is to say, P contains no intervals.

pf) Suppose P contains an interval (a, b) . $a < b$.

Then $S :=$ total length of removed intervals $\leq 1 - (b-a) < 1$.

$$\text{But } S = \sum_{n=1}^{\infty} 2^n 2^{-n} = \frac{1}{1-2/3} = 1 \quad \neq \square$$

4) Let $x \in [0, 1]$ & $x = \sum_{j=1}^{\infty} \frac{n_j}{3^j}$, $n_j \in \{0, 1, 2\}$.

If n_j is not terminating, the expression is unique,

but if $x = \frac{n_1}{3} + \dots + \frac{n_m}{3^m}$ ($n_m \neq 0$) we choose $n_m = 1$ or 2 .

< If $n_m = 1$, $\frac{1}{3^m} = \frac{2}{3^{m+1}} + \frac{2}{3^{m+2}} + \dots$

If $n_m = 2$, finite expansion. $\Rightarrow$ we choose $n_m = 1$.

then the expression becomes unique.

With this convention, let $x \in [0, 1]$ & $x = \sum_{j=1}^{\infty} \frac{n_j}{3^j}$.

then $x \in P$ iff $n_j \in \{0, 2\}$.

pf) By induction. $* P_{k+1} = \frac{1}{3}P_k \cup (\frac{2}{3} + \frac{1}{3}P_k)$.

($\Rightarrow$) k^{th} digit $\in \{0, 2\} \rightarrow (k+1)^{\text{th}}$ digit $\in \{0, 2\}$.

b) P is uncountable.

pf) $x \in P$. $x = \sum_{j=1}^{\infty} \frac{n_j}{3^j}$. ($n_j \in \{0, 2\}$). $f: P \rightarrow [0, 1]$.

$f(x) = \sum_{j=1}^{\infty} \frac{n_j}{2^j}$, then f is surjective from P to $[0, 1]$.

Remark. there are points in the Cantor P other than the endpoints of intervals.

ex. $\frac{1}{4} = 0.020202\dots$

$\hookrightarrow$ finite. for each P_n .

$\rightarrow$ countable.

$\rightarrow$ no intervals. but uncountably many.

§ 3.1. Convergent Sequences.

Def Let $\{p_n\}_{n=1}^{\infty}$ be a sequence in a metric space X .

We say $\{p_n\}_{n=1}^{\infty}$ **converges** if there is an $p \in X$ with the following property

: For every $\varepsilon > 0$, $\exists N(\varepsilon) \in \mathbb{N}$ s.t. $d(p_n, p) < \varepsilon$ if $n \geq N$.

In this case, $\lim_{n \rightarrow \infty} p_n = p$ or $p_n \rightarrow p$. ($\Leftrightarrow p_n \in N(\varepsilon)$)

Ex1 $a_n = \frac{n+1}{3n+2} \rightarrow$ claim: $\lim_{n \rightarrow \infty} a_n = \frac{1}{3}$.

Let $\varepsilon > 0$. Choose N s.t. $\frac{1}{N} < \varepsilon \Rightarrow |a_n - \frac{1}{3}| = \frac{1}{3(3n+2)} < \frac{1}{n} < \varepsilon$ if $n \geq N$.

Ex2 $\lim_{n \rightarrow \infty} \frac{1}{n^x} = 0$ ($x > 0$).

Ex3 $\{p_n\}_{n=1}^{\infty} \subset \mathbb{R}^k \rightarrow p_n = (x_n^1, \dots, x_n^k)$

$\{p_n\}$ converges iff $\{x_n^j\}_{n=1}^{\infty}$ converges for $1 \leq j \leq k$.

pf) ($\Rightarrow$) $\lim_{n \rightarrow \infty} p_n = a = (a^1, \dots, a^k)$. $\exists N$ s.t. $|p_n - a| < \varepsilon$ if $n \geq N$.
 $|x_n^j - a^j| < \varepsilon$ for $1 \leq j \leq k$.
 $|p_n - a| = \sqrt{(x_n^1 - a^1)^2 + \dots + (x_n^k - a^k)^2}$

($\Leftarrow$) $\lim_{n \rightarrow \infty} x_n^j = a^j$, $1 \leq j \leq k$ & $a^* = (a^1, \dots, a^k)$.

Let $\varepsilon > 0$. Since $x_n^j \rightarrow a^j$, $\exists N_j$ s.t. $|x_n^j - a^j| < \frac{\varepsilon}{\sqrt{k}}$ if $n \geq N_j$.

$\sqrt{(x_n^1 - a^1)^2 + \dots + (x_n^k - a^k)^2} < \varepsilon$
 $|p_n - a^*| = \left(\left(\frac{\varepsilon}{\sqrt{k}} \right)^2 \cdot k \right)^{1/2} < \varepsilon$ if $n \geq \max(N_1, \dots, N_k)$

Ex4 $X = \{f: f: [0,1] \rightarrow \mathbb{R}, \text{ bounded}\} \rightarrow$ i.e. $\sup_{x \in [0,1]} |f(x)| \leq M$ for some $M > 0$.

& $d(f, g) = \sup_{x \in [0,1]} |f(x) - g(x)|$.

$f_n(x) = x^n/n \rightarrow f_n \rightarrow 0$. ($\because d(f_n, 0) \leq \frac{1}{n}$)
 for $\forall x \in [0,1]$.

pf) Given $\varepsilon > 0$.

$\exists N$ s.t. $\sup_{x \in [0,1]} \left| \frac{x^n}{n} \right| < \varepsilon$ if $n \geq N$.
 $= \frac{1}{n}$.

Def A sequence $\{p_n\}_{n=1}^{\infty}$ in a metric space is said to be **bounded** if there are $p \in X$ & $M \geq 0$ s.t. $d(p_n, p) < M$ for $\forall n \in \mathbb{N}$.

Thm Let $\{p_n\}_{n=1}^{\infty} \subset X$.

1) If $\{p_n\}$ converges, then the limit is unique.

pf) Suppose $\lim_{n \rightarrow \infty} p_n = p$ & $\lim_{n \rightarrow \infty} p_n = p'$. Given $\varepsilon > 0$.

$\exists N$ s.t. $d(p_n, p) < \frac{\varepsilon}{2}$ & $d(p_n, p') < \frac{\varepsilon}{2}$ if $n \geq N$.

$d(p, p') \leq d(p_n, p) + d(p_n, p') < \varepsilon$ if $n \geq N$.

Since ε is arbitrary, $d(p, p') = 0$ $\square$

2) Every convergent sequence ($\{p_n\}$) is bounded. ($\Leftrightarrow X$ bounded $\Rightarrow X$ converge)

pf) $\lim_{n \rightarrow \infty} p_n = p$ for some $p \in X$.

$\exists N$ s.t. $d(p_n, p) < 1$ if $n \geq N$.

$d(p_n, p) < \max(d(p_1, p), \dots, d(p_{N-1}, p), 1) = M$ $\square$

3) Let $E \subset X$, if p is a limit point of E , then there is a sequence

$\{p_n\}$, $p_n \neq p$ such that $\lim_{n \rightarrow \infty} p_n = p$.

pf) Since p is a limit point of E , $\exists p_n \neq p \in E$ s.t. $d(p_n, p) < \frac{1}{n}$.

claim: $\lim_{n \rightarrow \infty} p_n = p$. Let $\varepsilon > 0$. choose N s.t. $\frac{1}{N} < \varepsilon$.

Then $d(p_n, p) < \varepsilon$ if $n \leq N$ $\square$

§ 3.2. Sequences in $\mathbb{R}$.

Thm Let $\{a_n\}, \{b_n\} \subset \mathbb{R}$. Suppose $\lim_{n \rightarrow \infty} a_n = a$ & $\lim_{n \rightarrow \infty} b_n = b$. Then

$$1) \lim_{n \rightarrow \infty} (a_n + b_n) = a + b.$$

$$2) \lim_{n \rightarrow \infty} a_n b_n = ab.$$

$$3) \text{ if } a \neq 0 \text{ & } a_n \neq 0, \lim_{n \rightarrow \infty} \frac{1}{a_n} = \frac{1}{a}. \quad \left. \vphantom{\lim_{n \rightarrow \infty} \frac{1}{a_n} = \frac{1}{a}} \right\} \text{Cor. } \lim_{n \rightarrow \infty} \frac{b_n}{a_n} = \frac{b}{a}.$$

pf 1) $\exists N$ s.t. $|a_n - a| < \frac{\varepsilon}{2}$ & $|b_n - b| < \frac{\varepsilon}{2}$ if $n \geq N$.

$$|a_n + b_n - (a + b)| \leq |a_n - a| + |b_n - b| < \varepsilon \text{ if } n \geq N.$$

2) Since $\{a_n\}, \{b_n\}$ are convergent, $\exists M$ s.t. $|a_n| \leq M$ & $|b_n| \leq M$.

Let $\varepsilon > 0$. $\exists N$ s.t. $|a_n - a| < \frac{\varepsilon}{2M}$ & $|b_n - b| < \frac{\varepsilon}{2M}$.

$$|a_n b_n - ab| = |a_n(b_n - b) + (a_n - a)b| \leq |a_n| |b_n - b| + |a_n - a| |b| < M \cdot \frac{\varepsilon}{2M} + \frac{\varepsilon}{2M} \cdot M = \varepsilon \text{ if } n \geq N.$$

3) Since $a_n \rightarrow a$, $\exists N$ s.t. $|a_n - a| < \frac{|a|}{2}$ if $n \geq N$.

Also, since $|a| \leq |a - a_n| + |a_n| < \frac{1}{2}|a| + |a_n|$, $|a_n| > \frac{|a|}{2}$ if $n \geq N$.

Given $\varepsilon > 0$. $\exists N$ s.t. $|a_n - a| < \frac{1}{2}|a|^2 \varepsilon$ if $n \geq N$.

$$\left| \frac{1}{a_n} - \frac{1}{a} \right| = \frac{|a - a_n|}{|a_n a|} < \frac{2}{|a|^2} |a_n - a| < \varepsilon \text{ if } n \geq N.$$

Thm. Let $\{a_n\}, \{b_n\} \subset \mathbb{R}$ if $\lim_{n \rightarrow \infty} a_n = 0$ & $\{b_n\}$ is bounded, then $\lim_{n \rightarrow \infty} a_n b_n = 0$.

pf) Given $\varepsilon > 0$.

$$\exists N \text{ s.t. } |a_n| < \frac{\varepsilon}{M} \text{ if } n \geq N. \text{ & } \exists M \text{ s.t. } |b_n| \leq M \text{ for } \forall n \in \mathbb{N}.$$

$$\exists N \text{ s.t. } |a_n b_n| \leq |a_n| M < \varepsilon \text{ if } n \geq N.$$

Thm Let $\{a_n\}, \{b_n\}, \{c_n\} \subset \mathbb{R}$. Suppose $a_n \leq c_n \leq b_n$ & $\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} b_n = a$.

Then $\lim_{n \rightarrow \infty} c_n = a$.

pf Given $\varepsilon > 0$. $\exists N$ s.t. $a - \varepsilon < a_n < a + \varepsilon$ & $a - \varepsilon < b_n < a + \varepsilon$ if $n \geq N$.

Then $a - \varepsilon < c_n < a + \varepsilon$ if $n \geq N$. i.e. $|c_n - a| < \varepsilon$ if $n \geq N$.

THEOREM 3.2.6 .

- (a) If $p > 0$, then $\lim_{n \rightarrow \infty} \frac{1}{n^p} = 0$.
- (b) If $p > 0$, then $\lim_{n \rightarrow \infty} \sqrt[p]{p} = 1$.
- (c) $\lim_{n \rightarrow \infty} \sqrt[n]{n} = 1$.
- (d) If $p > 1$ and α is real, then $\lim_{n \rightarrow \infty} \frac{n^\alpha}{p^n} = 0$.
- (e) If $|p| < 1$, then $\lim_{n \rightarrow \infty} p^n = 0$.
- (f) For all $p \in \mathbb{R}$, $\lim_{n \rightarrow \infty} \frac{p^n}{n!} = 0$.

§ 3.3. Monotone Sequences

Def Let $\{a_n\} \subset \mathbb{R}$. We say $\{a_n\}$ is

- 1) monotonic increasing if $a_n \leq a_{n+1}$. $\downarrow a_n$
 - 2) strictly increasing if $a_n < a_{n+1}$. $\downarrow a_n$
 - 3) monotonic decreasing if $a_n \geq a_{n+1}$. $\downarrow a_n$
 - 4) strictly decreasing if $a_n > a_{n+1}$. $\downarrow a_n$
- $\{a_n\}$ monotonic: converge $\Leftrightarrow$ bounded.

Thm Let $\{a_n\}$ be a monotonic increasing (decreasing) sequence. If $\{a_n\}$ is bounded above (below), then $\{a_n\}$ converges.

pf) Enough to prove monotone increasing case since $\lim_{n \rightarrow \infty} a_n = a \Rightarrow \lim_{n \rightarrow \infty} -a_n = -a$.
 $E = \{a_n, n=1, 2, \dots\}$. E is bounded & nonempty. Let $\alpha = \sup E$.
Let $\varepsilon > 0$. Since $\alpha = \sup E$, $\exists a_N$ s.t. $\alpha - \varepsilon < a_N$.
 $0 \leq \alpha - a_n \leq \alpha - a_N < \varepsilon$ if $n \geq N$.
 $\therefore |\alpha - a_n| < \varepsilon$ if $n \geq N$. $\square$

Def We say $\lim_{n \rightarrow \infty} a_n = \infty$ if for any $M > 0$, $\exists N$ s.t. $a_n > M$ if $n \geq N$.
 $\lim_{n \rightarrow \infty} a_n = -\infty$ if for any $M > 0$, $\exists N$ s.t. $a_n < -M$ if $n \geq N$.

Thm If $\{a_n\}$ monotonic increasing & not bounded, $\lim_{n \rightarrow \infty} a_n = \infty$.
(decreasing) $(\lim_{n \rightarrow \infty} a_n = -\infty)$

§ 3.4. Subsequences.

Def Let $\{a_n\}_{n=1}^{\infty} \subset X$ & $\{n_k\}_{k=1}^{\infty}$ s.t. $n_1 < n_2 < \dots$. Then $\{p_{n_k}\}_{k=1}^{\infty}$ is called a subsequence of $\{p_n\}_{n=1}^{\infty}$.

If $p \in X$ is a limit of subsequence, then p is called "a subsequential limit".

Ex. $p_n = (-1)^n$. $p_2 = p_4 = \dots = p_{2k} = 1 \Rightarrow$ increasing subsequence.

• $\mathbb{Q} = \{a_n\}_{n=1}^{\infty} \Rightarrow$ ~~does~~ subsequential limit.

Thm Let $\{p_n\}_{n=1}^{\infty} \subset X$. If $\{p_n\}_{n=1}^{\infty}$ converges to p , then every subsequence of $\{p_n\}_{n=1}^{\infty}$ converges to p .

($\Leftarrow$ If $\exists$ non-convergent subsequence or $\{p_{n_k}\}$ converges but not to p , then $\{p_n\}$ does not converge.)

Pr Let $\{p_{n_k}\}_{k=1}^{\infty}$ be a subsequence.

Let $\varepsilon > 0$. $\exists N$ s.t. $|p_n - p| < \varepsilon$ if $n \geq N$.

Let L be a number s.t. $n_L \geq N \Rightarrow |p_{n_k} - p| < \varepsilon$ if $k \geq L$.

Thm Let K be a cpe subset of X & $\{p_n\}_{n=1}^{\infty} \subset K$. Then $\{p_n\}_{n=1}^{\infty}$ has convergent subsequence (in K).

Pr Let $E = \{p_n : n=1, 2, \dots\}$.

1) If E is finite, then $\exists p \in E$ and $n_1 < n_2 < \dots$ s.t. $p_{n_k} = p \Rightarrow \lim_{k \rightarrow \infty} p_{n_k} = p$.

2) If E is infinite, E has a limit point $p \in K \Rightarrow$ subsequence select.

Since p is limit point of E , $\exists p_{n_1}$ s.t. $|p_{n_1} - p| < 1$.

p is a limit point of $\{p_n : n \geq n_1\}$. $\exists p_{n_2}$ s.t. $|p_{n_2} - p| < 1/2$ continue.

Then $n_1 < n_2 < \dots$ $|p_{n_k} - p| < 1/k$ for all k . $\Rightarrow \{p_{n_k}\}$ is convergent subsequence.

$\exists k_0 < \varepsilon$.

Cor If $\{p_n\}_{n=1}^{\infty} \subseteq \mathbb{R}^k$ is bounded, then $\{p_n\}_{n=1}^{\infty}$ has a convergent subsequence.

pf) Since $\{p_n\}_{n=1}^{\infty}$ is bounded, $\exists m > 0$ s.t. $\{p_n\}_{n=1}^{\infty} \subset \underbrace{[-m, m] \times \dots \times [-m, m]}_{k\text{-cell.}}$

Since $[-m, m] \times \dots \times [-m, m]$ is cpt, $\{p_n\}_{n=1}^{\infty}$ has a convergent ^{k-cell.} subsequence
 $\hookrightarrow$ closed & bounded.

§ 3.4. $\limsup$ & $\liminf$ of a sequence $(s_n) \in \mathbb{R}$

• Let $\{s_n\}_{n=1}^{\infty} \subseteq \mathbb{R}$. $\sup_{n \geq k} s_n$ ($k=1, 2, \dots$) = $\sup_{n \geq k} s_n$ is monotone decreasing.

Ex $a_n = n$. $\sup_{n \geq k} a_n = \infty$. / $a_n = -n$. $\sup_{n \geq k} a_n = k$.

• $\lim_{k \rightarrow \infty} \sup_{n \geq k} s_n \in \{+\infty, -\infty, \text{real number}\}$ & that. ($\therefore$ monotone seq.)

Def $\lim_{n \rightarrow \infty} \sup_{n \geq k} s_n = \lim_{k \rightarrow \infty} \sup_{n \geq k} s_n$ ($\overline{\lim}_{n \rightarrow \infty} s_n$)

$\lim_{n \rightarrow \infty} \inf_{n \geq k} s_n = \lim_{k \rightarrow \infty} \inf_{n \geq k} s_n$ ($\in \mathbb{R} \cup \{+\infty, -\infty\}$)

Remark 1) If $\{s_n\}$ is not bounded above, $\sup_{n \geq k} s_n = \infty \Rightarrow \lim_{n \rightarrow \infty} \sup_{n \geq k} s_n = \infty$.

2) If $\{s_n\}$ is bounded, $\lim_{n \rightarrow \infty} \sup_{n \geq k} s_n \in \mathbb{R}$.

3) $\lim_{n \rightarrow \infty} \inf_{n \geq k} s_n \leq \lim_{n \rightarrow \infty} \sup_{n \geq k} s_n$. (by def.)

Thm a) $\lim_{n \rightarrow \infty} \sup_{n \geq k} s_n = \beta \in \mathbb{R}$ iff for every $\varepsilon > 0$,

1) $\exists N$ s.t. $s_n < \beta + \varepsilon$ if $n \geq N$.

2) Given L . $\exists n \geq L$ s.t. $\beta - \varepsilon < s_n$.

b) $\lim_{n \rightarrow \infty} \sup_{n \geq k} s_n = \infty$ iff for any given $M & k$,

$\exists n \geq k$ such that $s_n \geq M$. ($\Leftrightarrow \sup_{n \geq k} s_n = \infty$).

c) $\lim_{n \rightarrow \infty} \sup_{n \geq k} s_n = -\infty$ iff $\lim_{n \rightarrow \infty} s_n = -\infty$.

($\Leftrightarrow$ Given $M > 0$, $\sup_{n \geq k} s_n < -M$ if $k \geq N$).

$\Rightarrow s_k < -M$ for $k \geq N$, $\lim_{n \rightarrow \infty} s_n = -\infty$.

Thm a) $\liminf_{n \rightarrow \infty} S_n = \beta \in \mathbb{R}$ iff for any $\varepsilon > 0$,

1) $\exists N$ s.t. $S_n > \beta - \varepsilon$ if $n \geq N$.

2) Given L , $\exists n \geq L$ s.t. $\beta + \varepsilon > S_n$.

b) $\liminf_{n \rightarrow \infty} S_n = -\infty$ iff for any given M & K

$\exists n \geq K$ s.t. $S_n \leq -M$.

c) $\liminf_{n \rightarrow \infty} S_n = \infty$ iff $\lim_{n \rightarrow \infty} S_n = \infty$.

Cor $\limsup_{n \rightarrow \infty} S_n = \liminf_{n \rightarrow \infty} S_n$ iff $\lim_{n \rightarrow \infty} S_n$ exists in $\mathbb{R} \cup \{-\infty, \infty\}$.

Pf ($\Leftarrow$) $\exists N$ s.t. $\alpha - \varepsilon < S_n < \alpha + \varepsilon$ if $n \geq N$.

($\Rightarrow$) If $\limsup_{n \rightarrow \infty} S_n = \liminf_{n \rightarrow \infty} S_n = \beta \in \mathbb{R}$, given $\varepsilon > 0$.

$\exists N_1$ s.t. $S_n < \beta + \varepsilon$ if $n \geq N_1$.

$\exists N_2$ s.t. $S_n > \beta - \varepsilon$ if $n \geq N_2$.

$|S_n - \beta| < \varepsilon$ if $n \geq \max(N_1, N_2)$

Thm Let $\{S_n\}_{n=1}^{\infty} \subset \mathbb{R}$ and E = the set of subsequential limits in $\mathbb{R} \cup \{-\infty, \infty\}$.

Then $\limsup_{n \rightarrow \infty} S_n = \sup E$, $\liminf_{n \rightarrow \infty} S_n = \inf E$.

Pf Hw

§ 3.6. Cauchy Sequence.

Def Let $\{p_n\}_{n=1}^{\infty}$ be a sequence in metric space (X, d) . We say $\{p_n\}_{n=1}^{\infty}$ is a Cauchy Sequence if

for every $\varepsilon > 0$, $\exists N$ s.t. $d(p_n, p_m) < \varepsilon$ if $n, m \geq N$.

Remark. Let $m > n$. Then $m = n + k$, $k \geq 1$.

$\Rightarrow d(p_{n+k}, p_n) < \varepsilon$ if $n \geq N$ for all $k = 1, 2, \dots$

$\lim_{n \rightarrow \infty} d(p_{n+k}, p_n) = 0$ if $\{p_n\}$ is Cauchy seq.
the converse is however false.

Thm Let $\{p_n\} \subset X$.

1) If $\{p_n\}$ is convergent, then $\{p_n\}$ is a Cauchy seq.
($\lim_{n \rightarrow \infty} p_n = p$)

pf) Given $\varepsilon > 0$. $\exists N$ s.t. $d(p_n, p) < \varepsilon/2$ if $n \geq N$.

$d(p_m, p_n) \leq d(p_m, p) + d(p_n, p) < \varepsilon$ if $m, n \geq N$.

2) If $\{p_n\}$ is a Cauchy seq., then $\{p_n\}$ is bounded.

pf) $\exists N$ s.t. $d(p_n, p_m) < 1$ if $n, m \geq N$.

In particular, $d(p_n, p_N) < 1$ if $n \geq N$.

Then, $d(p_n, p_N) \leq \max(d(p_1, p_N), \dots, d(p_{N-1}, p_N), 1)$

Thm* Let $\{p_n\}$ be a Cauchy seq. in X ,

If $\{p_n\}$ has a convergent subsequence in X , then $\{p_n\}$ converges in X .

Pt) Let $\{p_{n_k}\}_{k=1}^{\infty}$ be a subseq. of $\{p_n\}$ st. $\lim_{k \rightarrow \infty} p_{n_k} = p$ for some $p \in X$.

claim: $\lim_{n \rightarrow \infty} p_n = p$.

Given $\varepsilon > 0$. $\exists L$ s.t. $d(p_n, p_m) < \varepsilon/2$ if $n, m \geq L$.

$\exists l$ s.t. $d(p_{n_k}, p) < \varepsilon/2$ if $k \geq l$.

Choose $m \geq \max(L, n_l)$. then $n_m > n_l, L$.

$d(p_n, p) \leq d(p_n, p_{n_m}) + d(p_{n_m}, p) < \varepsilon$ if $n \geq \max(L, n_l)$. □

Cor. Let $\{p_n\}$ be a Cauchy seq. in X . If $\{p_n\}$ is a subset of a cpt set, then $\{p_n\}$ converges.

$\Rightarrow$ In particular, if $\{p_n\}$ is a Cauchy seq. in $\mathbb{R}^k$, $\{p_n\}$ converges.

Def We say a metric space X is complete if every Cauchy sequence in X converges in X .
 $\rightarrow$ bounded $\Rightarrow$ cpt.
 $\rightarrow \exists$ conv subseq.

Ex 1) $\mathbb{R}^k$ is complete metric space.

2) $\mathbb{Q} \subset \mathbb{R}$ is not complete ($\because \exists \sqrt{2} \in \mathbb{R} \setminus \mathbb{Q}$).

Def Let $\{p_n\} \subset X$. We say $\{p_n\}$ is contractive if $d(p_{n+1}, p_n) \leq c d(p_n, p_{n-1})$ ($\forall n \geq 2$), for some constant $0 < c < 1$.

$\Rightarrow d(p_{n+1}, p_n) \leq c \cdot d(p_n, p_{n-1}) \leq c^2 d(p_{n-1}, p_{n-2}) \leq \dots \leq c^{n-1} d(p_2, p_1)$.

$m = n+k$, $d(p_{n+k}, p_n) \leq d(p_{n+k}, p_{n+k-1}) + d(p_{n+k-1}, p_{n+k-2}) + \dots + d(p_{n+1}, p_n)$
 $\leq c^{n+k-2} d(p_2, p_1) + c^{n+k-3} d(p_2, p_1) + \dots + c^{n-1} d(p_2, p_1)$
 $\leq c^{n-1} d(p_2, p_1) (c^{k-1} + \dots + 1) \leq c^{n-1} d(p_2, p_1) / (1-c)$

$\therefore d(p_{n+k}, p_n) \leq c^{n-1} \cdot \frac{d(p_2, p_1)}{1-c} \therefore \{p_n\}_{n=1}^{\infty}$ is a Cauchy seq.

Thm If $\{p_n\}$ is a contractive sequence in a complete metric space, then $\{p_n\}$ converges

Ex $a_n = 2^{-n} \rightarrow c = \frac{1}{2}$.

§ 3.7. Series of Real Numbers.

Def Let $\{a_n\}_{n=1}^{\infty}$ be a sequence. Set $S_n = \sum_{k=1}^n a_k$ (n -th partial sum)

• We say $\sum_{n=1}^{\infty} a_n$ converges iff $\{S_n\}$ converges.

($\Rightarrow \forall \varepsilon, \exists N$ s.t. $|S_m - S_n| < \varepsilon$ if $n, m \geq N$).

Thm $\sum_{n=1}^{\infty} a_n$ converges iff $\forall \varepsilon > 0, \exists N$ s.t. $|\sum_{k=n+1}^m a_k| < \varepsilon$ if $n, m \geq N$.

A) In particular, $|a_{n+1}| < \varepsilon$ if $n \geq N \Rightarrow \lim_{n \rightarrow \infty} a_n = 0$.

B) $\sum_{k=1}^{\infty} \frac{1}{k}$. $S_{2n} - S_n = \frac{1}{2n+1} + \dots + \frac{1}{n+1} \geq \frac{1}{2n+1} + \dots + \frac{1}{2n} = \frac{1}{2n} \cdot n = \frac{1}{2}$.
 $\Rightarrow \{S_n\}$ is not a Cauchy seq. $\Rightarrow \sum 1/k$ not converges.

C) Suppose a_n is a decreasing seq. & $a_n \geq 0$.

Then $\sum_{n=1}^{\infty} a_n$ converges iff $\sum_{n=1}^{\infty} 2^n a_{2^n}$ converges.

Ex $\sum_{k=1}^{\infty} \frac{1}{k^2} \rightarrow \sum_{n=1}^{\infty} 2^n \cdot \frac{1}{2^{2n}} = \sum_{n=1}^{\infty} \frac{1}{2^n} = 1$ converges.

pf Note. $S_n = \sum_{k=1}^n a_k$ converges iff S_n is bounded.

$$(\Leftarrow) \underbrace{a_1 + a_2 + a_3 + a_4}_{2^1} + \underbrace{a_5 + a_6 + a_7 + a_8}_{2^2} + \underbrace{a_9 + \dots + a_{16}}_{2^3} + \dots$$

$$\leq a_1 + 2^1 \cdot a_2 + 2^2 \cdot a_4 + \dots = a_1 + \sum_{n=1}^{\infty} 2^n a_{2^n}.$$

Ch4. Limits and Continuity

§ 4.1. Limit of a function

Def (Limit) Let (X, d) be a metric space, $E \subset X$ and $f: E \rightarrow \mathbb{R}(\mathbb{C})$

Suppose p is a limit point of E , we write $\lim_{x \rightarrow p} f(x) = L$

if there is an $L \in \mathbb{R}$ satisfying the following:

for every $\varepsilon > 0$, $\exists \delta > 0$ s.t. $|f(x) - L| < \varepsilon$ if $0 < d(x, p) < \delta$ and $x \in E$.
($d(x, p) < \delta \Rightarrow |f(x) - L| < \varepsilon$)

Remark 1) $\lim_{x \rightarrow p} f(x)$ is independent of $f(p)$. (not required that $p \in E$)

2) $\lim_{x \rightarrow p} f(x)$ is unique (if exists)

fb) Suppose $\lim_{x \rightarrow p} f(x) = L$ and $\lim_{x \rightarrow p} f(x) = L'$.

Given $\varepsilon > 0$. $\exists \delta$ s.t. $|f(x) - L| < \varepsilon/2$ if $0 < d(x, p) < \delta$. $x \in E$.

$\exists \delta'$ s.t. $|f(x) - L'| < \varepsilon/2$ if $0 < d(x, p) < \delta'$. $x \in E$.

Let $x \in E$ such that $0 < d(x, p) < \min(\delta, \delta')$

$$|L - L'| \leq |f(x) - L'| + |f(x) - L| < \varepsilon \Rightarrow |L - L'| < \varepsilon \Rightarrow L = L'$$

3) $f: [0, 1] \rightarrow \mathbb{R}$, $f(x) = \begin{cases} 0 & \text{if } x \in \mathbb{Q} \\ x & \text{if } x \notin \mathbb{Q} \end{cases}$

$$\rightarrow \lim_{x \rightarrow 0} f(x) = 0.$$

if $x \neq 0$, $\lim_{x \rightarrow x} f(x)$ is arbitrary.

4) Limit is meaningless if not limit point. (= isolated point).

(e.g. $\lim_{x \rightarrow 1} x^2 = 1$ is meaningless if $x=1$ is isolated point...)

Thm Let $E \subset X$. $f: E \rightarrow \mathbb{R}$, p is a limit point of E .

$\lim_{x \rightarrow p} f(x) = L$ iff $\lim_{n \rightarrow \infty} f(p_n) = L$ for every seq. $\{p_n\} \subset E$, $p_n \neq p$ such that $p_n \rightarrow p$ as $n \rightarrow \infty$.

pf) ($\Rightarrow$) Let $\{p_n\} \subset E$ a sequence s.t. $p_n \neq p$ & $p_n \rightarrow p$.

Given $\varepsilon > 0$. $\exists \delta > 0$ s.t. $|f(x) - L| < \varepsilon$ if $0 < d(x, p) < \delta$. $x \in E$.

Let $N \in \mathbb{N}$ s.t. $0 < d(p_n, p) < \delta$ if $n \geq N$.

$\therefore |f(p_n) - L| < \varepsilon$ if $n \geq N$.

($\Leftarrow$) (Contrapositive) Suppose $\lim_{x \rightarrow p} f(x) \neq L$. then for some $\varepsilon > 0$,

there is no $\delta > 0$ s.t. $|f(x) - L| < \varepsilon$ even if $0 < d(x, p) < \delta$. $x \in E$.

($\Leftarrow$ every $\delta > 0$, $\exists x$ s.t. $|f(x) - L| \geq \varepsilon$ and $0 < d(x, p) < \delta$, $x \in E$).

In particular, taking $\delta = 1/n$, $\exists p_n \in E$ s.t. $|f(p_n) - L| \geq \varepsilon$, and $0 < d(p_n, p) < 1/n$, $p_n \in E$.

$\lim_{n \rightarrow \infty} p_n = p$, $p_n \in E$, but $\lim_{n \rightarrow \infty} f(p_n) \neq L$. $\nexists$

Thm Let $E \subset X$. $f, g: E \rightarrow \mathbb{R}$ & p is a limit point of E . $\lim_{x \rightarrow p} f(x) = A$. $\lim_{x \rightarrow p} g(x) = B$.

1) $\lim_{x \rightarrow p} (f(x) + g(x)) = A + B$.

2) $\lim_{x \rightarrow p} f(x)g(x) = AB$.

3) if $B \neq 0$, $\lim_{x \rightarrow p} \frac{1}{g(x)} = \frac{1}{B}$.

$|B| \leq |B - g(x)| + |g(x)| < \frac{|B|}{2} + |g(x)|$

pf) $\exists \delta > 0$ s.t. $|g(x) - B| < \frac{|B|}{2}$ if $d(x, p) < \delta$. $x \in E$.

Given $\varepsilon > 0$. $\exists \delta' > 0$ s.t. $|g(x) - B| < \frac{|B|^2}{2} \varepsilon$ if $d(x, p) < \delta'$. $x \in E$. if $d(x, p) < \delta$, $x \in E$.

$\left| \frac{1}{g(x)} - \frac{1}{B} \right| = \frac{|B - g(x)|}{|g(x)||B|} \leq \frac{2}{|B|^2} |B - g(x)| < \varepsilon$ if $d(x, p) < \delta'$, $x \in E$.

Def (Limit at ∞) Let $E \subset \mathbb{R}$. $f: E \rightarrow \mathbb{R}$ such that

$E \cap (a, \infty) \neq \emptyset$ for all $a > 0$. Then
($(-\infty, a) \neq \emptyset$)

we write $\lim_{\substack{x \rightarrow \infty \\ (x \rightarrow -\infty)}} f(x) = L$ if there is an $L \in \mathbb{R}$ satisfying
: every $\varepsilon > 0$, $\exists M > 0$ s.t. $|f(x) - L| < \varepsilon$ if $x \geq M$, $x \in E$.
($x \leq -M$)

Ex $f(x) = \frac{1}{x}$. $E = \mathbb{R} \setminus \{0\}$. $\rightarrow \lim_{x \rightarrow \infty} f(x) = 0$.

$\rightarrow$ Let $\varepsilon > 0$ be given. Choose $M > 0$ such that $\frac{1}{M} < \varepsilon$.

$$|f(x) - 0| = \frac{1}{|x|} < \varepsilon \text{ if } x \geq M.$$

§ 4.2. Continuous Functions.

Def Let $E \subset X$. $f: E \rightarrow \mathbb{R}(\infty)$ and $p \in E$. We say

f is continuous at p if for every $\epsilon > 0$, there's $\delta > 0$ st.
 $|f(x) - f(p)| < \epsilon$ if $d(x, p) < \delta$. $x \in E$.

Remark: 1) If p is a limit point of E , $f \in \mathbb{R}$ continuous at p iff $\lim_{x \rightarrow p} f(x) = f(p)$

2) If p is an isolated point of E , any f is continuous at p .

$$\underline{\text{Ex}}_3) f: [0, 1] \rightarrow \mathbb{R}, \quad f(x) = \begin{cases} 0 & x \in \mathbb{Q} \\ x & x \notin \mathbb{Q} \end{cases}$$

$\rightarrow f$ is continuous at 0 ($\because f(0) = \lim_{x \rightarrow 0} f(x) = 0$)

f is discontinuous at $x \notin \mathbb{Q}$ ($\because$ ~~정수~~ ~~비정수~~ $\frac{d}{dx}$).

4) $f: [0, 1] \rightarrow \mathbb{R}$ $f = \begin{cases} x & \text{if } x \notin \mathbb{Q} \\ \frac{1}{n} & \text{if } x = \frac{m}{n} \text{ (in lowest terms)} \end{cases}$

$\rightarrow f$ is discontinuous at x if $x \in \mathbb{Q}$ (rational numbers)
 but f is continuous at x if $x \notin \mathbb{Q}$. (irrational numbers)

pf) Let $p \in \mathbb{Q}$ & $p \in [0, 1]$. Given $\varepsilon > 0$. Choose N st. $\frac{1}{N} < \varepsilon$.

$$S = \left\{ \frac{m}{a} \in [0, 1] : m, n \text{ has no common factor} \right\}$$
$$\delta = \min_{s \in S} |p-s| \neq 0 \quad (\because p \notin \mathbb{Q})$$

if $|x - p| < \delta$, $|f(x) - f(p)| = |f(x)| \leq \frac{1}{\sqrt{x}} < \varepsilon$

↪ $\begin{cases} \text{if } x \text{ is irrational} \rightarrow 0 \\ x \text{ is rational} \rightarrow \frac{1}{x^2} \text{ 이 됨.} \end{cases}$