<수리통계 2>

8장. 추정량의 비교

8.1. 추정량 비교의 기준.

8.1.1. 평균제곱오차 (MSE).

__Def__ $X_1 \dots X_n \sim f(x;\theta)$ $\quad \eta(\theta) \xleftarrow{\text{추정}} \hat\eta(X_1 \dots X_n) \doteq \hat\eta(X)$

$$E_\theta\left[(\hat\eta(X_1\dots X_n) - \eta(\theta))^2\right] = MSE(\hat\eta, \theta)$$

(다차원) $E_\theta\left[\|\hat\eta(X_1\dots X_n) - \eta(\theta)\|^2\right] = MSE(\hat\eta,\theta)$

__Note.__
$$MSE = E\Big[\big(\underbrace{\hat\eta(X) - E_\theta(\hat\eta(X))}_{A} + \underbrace{E_\theta(\hat\eta(X)) - \eta(\theta)}_{B}\big)^2\Big]$$

$$= E\left[(\hat\eta(X) - E_\theta(\hat\eta(X)))^2\right] + E\left[(E(\hat\eta(X)) - \eta(\theta))^2\right]$$

$$+ 2\,\underbrace{E\left[(\hat\eta(X) - E(\hat\eta(X)))\,B\right]}_{\to 0.}$$

$$= Var(\hat\eta(X)) + [\,E(\hat\eta(X)) - \eta(\theta)\,]^2 \quad (bias)^2$$

(다차원) $MSE = E\left[(\hat\eta(X) - \eta(\theta))^T(\hat\eta(X) - \eta(\theta))\right]$

$$= tr\left\{E\left[(\hat\eta(X) - \eta(\theta))(\hat\eta(X) - \eta(\theta))^T\right]\right\}$$

$$= tr\Big\{E\Big[\big(\underbrace{\hat\eta(X) - E(\hat\eta(X))}_{A} + \underbrace{E(\hat\eta(X)) - \eta(\theta)}_{B}\big)\big(\;\;''\;\;\big)^T\Big]\Big\}$$

$$= tr\left\{E[AA^T] + E[A]B^T + BE[A^T] + BB^T\right\}$$

$$= tr\{Var(\hat\eta(X))\} + \|bias(\hat\eta(X), \eta(\theta))\|^2.$$

8.1.2. 예제들

비교 가능 (예 8.1.1.) $X_1 \dots X_n \sim U(0,\theta)$. $\Rightarrow \hat\theta^{MLE} = X_{(n)}$, $\hat\theta^{MME} = 2\bar X$.

sol) • $MSE(\hat\theta^{MLE}, \theta) = E_\theta\left[(X_{(n)} - \theta)^2\right] = E_\theta\left[X_{(n)}^2 - 2\theta X_{(n)} + \theta^2\right]$

$\Rightarrow P(X_{(n)} \le x) = (P(X_i \le x))^n = (\frac{x}{\theta})^n.$

$pdf_{X_{(n)}}(x) = \frac{n}{\theta^n} x^{n-1}$

$$\Rightarrow E[X_{(n)}^2] = \int_0^\theta \frac{n}{\theta^n} x^{n+1}\,dx = \frac{n}{\theta^n}\frac{1}{n+2} x^{n+2}\Big|_0^\theta = \frac{n}{n+2}\theta^2.$$

$$E[X_{(n)}] = \int_0^\theta \frac{n}{\theta^n} x^n\,dx = \frac{n}{\theta^n}\frac{1}{n+1} x^{n+1}\Big|_0^\theta = \frac{n}{n+1}\theta$$

$$MSE = \frac{n}{n+2}\theta^2 - 2\theta^2\frac{n}{n+1} + \theta^2 = \theta^2\left(\frac{n}{n+2} - \frac{2n}{n+1} + 1\right) = \frac{n(n+1) - 2n(n+2) + (n+2)(n+1)}{(n+2)(n+1)}$$

$$= \frac{2}{(n+1)(n+2)}\theta^2$$

• $MSE(\hat\theta^{MME}, \theta) = E_\theta\left[(2\bar X - \theta)^2\right] = Var(2\bar X) + (E(2\bar X) - \theta)^2$

$$= 4\cdot Var(\bar X) + (2E(\bar X) - \theta)^2$$

$\bar X = \frac{1}{n}\sum X_i$, $\quad Var(\bar X) = \frac{1}{n}Var(X_i) = \theta^2/12n$

$$E(\bar X) = E(X) = \frac{\theta}{2}$$

$$MSE = 4\cdot\frac{\theta^2}{12n} = \frac{\theta^2}{3n}.$$

$\rightarrow$ MLE가 더 빨리 0으로 수렴.

(예 8.1.2.) $X_1 \dots X_n \sim N(\theta, 1) \rightarrow \begin{cases} \hat{\theta}^{MLE} = \bar{X}. \\ \hat{\theta}_H = \begin{cases} 0 & |\bar{X}| \le 1.96\frac{1}{\sqrt{n}} \\ \bar{X} & |\bar{X}| > 1.96\frac{1}{\sqrt{n}}. \end{cases} \end{cases}$

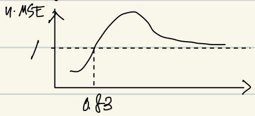
sol) • $MSE(\hat{\theta}^{MLE}, \theta) = E[(\bar{X}-\theta)^2] = \frac{1}{n}.$ ($\because$ bias $= 0$).

• $MSE(\hat{\theta}^H, \theta) = E[(\hat{\theta}^H - \theta)^2]$ ※ $\sqrt{n}\bar{X} = \sqrt{n}(\bar{X}-\theta) + \sqrt{n}\theta \overset{d}{=} Z + \sqrt{n}\theta.$

$= E[\theta^2 I(|Z+\sqrt{n}\theta| < 1.96)] + E[(\bar{X}-\theta)^2 I(|Z+\sqrt{n}\theta| > 1.96)]$

$= \theta^2 \cdot P_\theta(|Z+\sqrt{n}\theta| < 1.96) + \frac{1}{n} E_\theta(Z^2 I(|Z+\sqrt{n}\theta| > 1.96)).$

$\Rightarrow \sqrt{n}|\theta|$에 의존.



(표본 크기 작으면 $\hat{\theta}^H$이 더 나을수도..)

8.1.3. 최대평균제곱오차. 베이지안 평균제곱오차.

(a) minimax MSE estimator : $\underset{\hat{\eta}}{\operatorname{argmin}} \left(\max_{\theta \in \Omega} MSE_\theta [(\hat{\eta} - \eta(\theta))^2] \right)$

(b) 베이즈 추정량 : 모수에 대한 사전분포 $\pi(\theta)$.

$\rightarrow r(\pi, \hat{\eta}) = \int_\Omega E[(\hat{\eta}(X) - \eta(\theta))^2] \pi(\theta) d\theta.$

$\hat{\eta}^{Bayes}(X) = \underset{\hat{\eta}}{\operatorname{argmin}}\ r(\pi, \hat{\eta}).$

8.1.4. 평균제곱 상대오차 : $E\left[\left(\frac{\hat{\eta}(X)}{\eta(\theta)} - 1\right)^2\right].$

(예 8.1.3.) $X_1 \dots X_n \sim N(\mu, \sigma^2).$

모분산에 대한 추정량 : $\hat{\sigma}^2_c = c\sum(X_i - \bar{X})^2.$

$\Rightarrow \underset{c}{\operatorname{argmin}} \left(\max_{\theta \in \Omega} E[(\hat{\sigma}^2_c/\sigma^2 - 1)^2] \right) = \frac{1}{n+1}.$

pf) $E[(\hat{\sigma}^2_c/\sigma^2 - 1)^2] = E[(cV-1)^2].$ $V \sim \chi^2(n-1).$ $E(V) = n-1$, $Var(V) = 2(n-1).$

$= E[(c(V-(n-1)) + c(n-1) - 1)^2]$ ※

$a = c$, $b = c(n-1)-1$, $V-(n-1) = Y.$

$E[(aY+b)^2] = a^2 Var(Y) + b^2$ ($\because E(Y) = 0$).

$= \underbrace{2(n-1)}_{Var(Y)} \underbrace{c^2}_{a^2} + \underbrace{((n-1)c-1)^2}_{b^2} = \max E(\cdot)$

8.1.5. 전역최소분산불편 추정량 : UMVUE.

Def (a) $E_\theta[\hat{\eta}^*(X)] = \eta(\theta)$

(b) $\forall \hat{\eta}(X)$ s.t. $E_\theta[\hat{\eta}(X)] = \eta(\theta)$, $Var_\theta(\hat{\eta}^*(X)) \le Var_\theta(\hat{\eta}(X))$ $\forall \theta \in \Omega.$

$\Rightarrow \hat{\eta}^*$ is UMVUE of $\eta = \eta(\theta).$

Thm 8.1.1. (UMVUE의 유일성). $\hat{\eta}_1, \hat{\eta}_2$가 η의 UMVUE $\Rightarrow P_\theta(\hat{\eta}_1 = \hat{\eta}_2) = 1.$ $\forall \theta \in \Omega.$

pf). (i) Let $\hat{\eta}_3 = (\hat{\eta}_1 + \hat{\eta}_2)/2. \Rightarrow E[\hat{\eta}_3] = \eta(\theta).$ $\hat{\eta}_3$ is UE.

$\rightarrow$ (b)에 의해 $Var(\hat{\eta}_3) \ge Var(\hat{\eta}_1) = Var(\hat{\eta}_2).$

$\Leftrightarrow \quad Var(\hat\eta_3) = Var\left(\frac{1}{2}(\hat\eta_1+\hat\eta_2)\right) = \frac{1}{4}\{Var(\hat\eta_1)+Var(\hat\eta_2)+2Cov(\hat\eta_1,\hat\eta_2)\} \geq Var(\hat\eta_1)=Var(\hat\eta_2).$

$\Leftrightarrow \quad Cov(\hat\eta_1, \hat\eta_2) \geq Var(\hat\eta_1) = Var(\hat\eta_2). \quad \cdots (*)$

(2) $Var(\hat\eta_1-\hat\eta_2) = Var(\hat\eta_1) + Var(\hat\eta_2) - 2Cov(\hat\eta_1,\hat\eta_2) \leq 0. \Rightarrow$ '=' 만 가능.

(3) (분산 0의 의미) $\Rightarrow P_\theta(|\hat\eta_1-\hat\eta_2| < \varepsilon) \leq \frac{1}{\varepsilon^2} Var(\hat\eta_1-\hat\eta_2) = 0.$

$\Leftrightarrow P(\hat\eta_1 = \hat\eta_2) = 1..$

8.2. 충분통계량

8.2.1. 정의

통계량 $\quad$ $u(X)$로 표기.

<u>Def</u> $X_1 \dots X_n \sim f(x\cdot\theta)$, $Y = u(X_1\dots X_n)$, $\forall A \in \mathbb{R}^n.$

$P_\theta((X_1\cdots X_n)\in A \mid Y=y)$ not dependent on θ. 이면 Y: s.s. for $\theta\in\Omega$.

8.2.2. 분해 정리.

<u>Thm 8.2.1.</u> $\exists k_1, k_2 \in F$ s.t. $\prod_{i=1}^n f(x_i:\theta) = k_1(u(X),\theta)\, k_2(X_1\dots X_n)$

$\Leftrightarrow$ $Y = u(X_1\dots X_n)$ 이 θ의 s.s.

pf) 생략.

8.2.3. 예제들

<u>Ex 8.2.3.</u> $X_1 \dots X_n \sim Poi(\theta)$. $\theta > 0$

sol) $\prod_{i=1}^n f(x_i;\theta) = \prod_{i=1}^n \frac{e^{-\theta}\theta^{x_i}}{x!}$

$= e^{-n\theta}\,\theta^{x_1+\cdots+x_n} \cdot \prod_{i=1}^n \left(\frac{1}{x!}\right). \quad \Rightarrow \sum X_i$: s.s. of θ

<u>Ex 8.2.4.</u> $X_1\dots X_n \sim Exp(\theta)$. $\theta>0.$

sol) $\prod_{i=1}^n f(x_i;\theta) = \prod_{i=1}^n \frac{1}{\theta}\exp\left(-\frac{x_i}{\theta}\right) I(x_i>0).$

$= \left(\frac{1}{\theta}\right)^n \cdot \exp\left(-\frac{\sum X_i}{\theta}\right) \cdot \prod_{i=1}^n I(x_i>0) \quad \Rightarrow \sum X_i$: s.s. of θ.

<u>Ex 8.2.5</u> $X_1\dots X_n \sim Gamma(\alpha,\beta)$, $\alpha,\beta>0$.

sol) $\prod_{i=1}^n f(x_i:\theta) = \prod_{i=1}^n \frac{1}{\Gamma(\alpha)\beta^\alpha} x_i^{\alpha-1} e^{-x_i/\beta} I(x_i>0).$

$= \left(\frac{1}{\Gamma(\alpha)\beta^\alpha}\right)^n \left(\prod x_i\right)^{\alpha-1} \exp\left(-\sum X_i/\beta\right) \cdot \prod_{i=1}^n I(X_i>0).$

$\rightarrow \left(\prod X_i, \sum X_i\right)$: s.s. of (α,β).

Ex 8.2.2. $X_1 \dots X_n \sim \text{Bernoulli}(\theta)$. $0<\theta<1$.

sol) [Def. 이용]. $Y = X_1 + \dots + X_n \sim Bin(n, \theta)$.

$$P_\theta(X_1 = x_1, \dots X_n = x_n \mid X_1 + \dots + X_n = y) = \frac{\prod \theta^{x_i}(1-\theta)^{1-x_i} I(x_1 + \dots + x_n = y)}{\binom{n}{y}\theta^y (1-\theta)^{n-y}}$$

$$= \binom{n}{y}^{-1} I(x_1 + \dots + x_n = y), \quad \forall \theta \in (0,1)$$

$\Rightarrow$ not depend on θ, $Y = \sum X_i$ is s.s. of θ.

8.2.4. 다중모수 지수족에서의 충분통계량

Thm 8.2.2. $f(x;\theta) = \exp\{\sum_{j=1}^{k} g_j(\theta) T_j(x) - A(g(\theta)) + S(x)\}$. 꼴로 주어지고,

분포의 토대 $\mathcal{X} = \{x : f(x;\theta) > 0\}$ 이 모수에 의존하지 X.

$\Rightarrow \sum T(X_i) = \left(\sum T_1(X_i), \dots, \sum T_k(X_i)\right)^T$: s.s. of $\theta \in \Omega$.

pf). 분해정리 이용.

Thm 8.2.3. (충분통계량의 1-1 함수). $X_1 \dots X_n \sim f(x;\theta)$, $Y = u(X_1 \dots X_n)$ 이 θ의 s.s.

$W = g(Y)$. g는 1-1 함수 $\Rightarrow$ W는 θ의 s.s.

pf) From Def. $W = g(Y)$, $Y = g^{-1}(W)$.

8.2.5. 예제들

Ex 8.2.6. $X_1 \dots X_n \sim N(\mu, \sigma^2)$.

sol) $f(x;\mu,\sigma^2) = \frac{1}{\sqrt{2\pi}\sigma} \exp\{-\frac{(x-\mu)^2}{2\sigma^2}\}$.

$= \exp\{\frac{\mu}{\sigma^2}\underline{x} - \frac{1}{2\sigma^2}\underline{x^2} - \frac{\mu^2}{2\sigma^2} - \frac{1}{2}\log(2\pi\sigma^2)\}$. $\rightarrow$ 지수족.

$\Rightarrow (\sum X_i, \sum X_i^2)$: (μ, σ^2) 의 s.s.

$(\bar{X}, \frac{1}{n-1}\sum(X_i - \bar{X})^2)$: 〃

Ex 8.2.7. $X_1 \dots X_n \sim Beta(\alpha, 1)$. $\alpha > 0$.

sol) $f(x;\alpha) = \alpha x^{\alpha-1} I(0<x<1)$.

$= \exp\{(\alpha-1)\log x + \log\alpha\} I(0<x<1)$. $\rightarrow$ 지수족

$\Rightarrow$ $\sum \log X_i$: α의 s.s.

$\frac{1}{n}\sum \log X_i = \overline{\log X}$.

Ex 6.2.8. $X_1 \dots X_n \sim f$. (두개의 모수 지수분포). $\mu \in \mathbb{R}$, $\sigma > 0$.

sol) $f(x ; \mu, \sigma) = \frac{1}{\sigma} \exp\left(\frac{x-\mu}{\sigma}\right) I(\underbrace{x \geq \mu})$ → 지수족 적용 불가.

분해정리

$$\prod_{i=1}^{n} f(x_i ; \mu, \sigma) = \left(\frac{1}{\sigma}\right)^n \exp\left\{\frac{\sum X_i - n\mu}{\sigma}\right\} I(X_{(1)} \geq \mu).$$

$$= k_1(X_1 + \dots + X_n,\ X_{(1)} ; \mu, \sigma) \times k_2(\underbrace{X_1 \dots X_n}_{=1}).$$

$\Rightarrow (\sum X_i,\ X_{(1)})$: (μ, σ)의 s.s.

$(\sum (X_i - X_{(1)}),\ X_{(1)})$. 〃

Ex 6.2.9. $X_1 \dots X_n \sim U[\theta_1, \theta_2]$, $-\infty < \theta_1 < \theta_2 < \infty$.

sol) $\prod_{i=1}^{n} f(x_i ; \theta_1, \theta_2) = \prod_{i=1}^{n}\left(\frac{1}{\theta_2 - \theta_1} \cdot I(\theta_1 \leq x_i \leq \theta_2)\right)$

분해정리.

$$= \left(\frac{1}{\theta_2 - \theta_1}\right)^n \cdot I(\theta_1 \leq X_{(1)},\ X_{(n)} \leq \theta_2).$$

$\Rightarrow (X_{(1)}, X_{(n)})$: (θ_1, θ_2)의 s.s.

+) $X_i \sim U(-\theta, \theta)$

$\Rightarrow \max(|X_{(1)}|, |X_{(n)}|)$: s.s. for θ.

6.2.6. MLE of S.S.

Thm 6.2.4. $X_1 \dots X_n \sim f(x ; \theta)$.

$\hat{\theta}^{MLE}$ unique 하다면, $\hat{\theta}^{MLE}$는 S.S $S(X_1 \dots X_n)$의 함수이다.

pf) $L(\theta) = \prod f(x_i ; \theta) = k_1(S, \theta)\, k_2(X_1 \dots X_n)$.

$\hat{\theta}^{MLE} = \underset{\theta}{\operatorname{argmax}}\ k_1(S(X_1 \dots X_n), \theta)$.

Note • Minimal S.S. : Gamma (α, β)의 s.s.

$S_0 = (\prod X_i, \sum X_i)$.

$S_1 = (X_1 \dots X_n)$

$S_2 = (X_1, \prod_{i=2}^{n} X_i, \sum_{i=2}^{n} X_i)$

$S_3 = (X_1, X_2, \dots)$

→ $S_0 = h(S_i)$.

• Cauchy $(\theta, 1)$의 (minimal) s.s. : $X_{(1)} \dots X_{(n)}$.

• $X_1 \dots X_n \sim f(x ; \theta) \Rightarrow X_{(1)} \dots X_{(n)}$은 s.s. for θ.

pf) $P((X_1 \dots X_n) = x \mid X_{(1)} = y_1 \dots X_{(n)} = y_n) = \begin{cases} \frac{1}{n!} & x_1 = y_1, \dots, x_n = y_n. \\ 0 & \text{o.w.} \end{cases}$

→ not depend on θ.

8.2.17. Rao–Blackwell 정리.

Thm (8.2.4.) $X_1 \dots X_n \sim f(x;\theta)$. $Y = u(X_1 \dots X_n)$: s.s. for θ, $\eta(\theta)$의 추정량 $\hat{\eta}$

$$\hat{\eta}^{RB}(X_1 \dots X_n) = E[\hat{\eta}(X_1 \dots X_n) \mid Y = y]$$

$$\Rightarrow MSE(\hat{\eta}^{RB}, \eta(\theta)) \le MSE(\hat{\eta}, \eta(\theta))$$

(i) $E(\hat{\eta}^{RB}) = E(\hat{\eta})$
(ii) $Var(\hat{\eta}^{RB}) \le Var(\hat{\eta})$.

pf) $Var_Y(Y) = Var_X(E(Y|X)) + E_X(Var(Y|X))$.

(ii) $Var(\hat{\eta}) = Var(E(\hat{\eta}|Y)) + E(\underbrace{Var(\hat{\eta}|Y)}_{\ge 0})$

$\ge Var(E(\hat{\eta}|Y)) = Var(\hat{\eta}^{RB})$.

(i) $E[\hat{\eta}] = E[E(\hat{\eta}|Y)] = E[\hat{\eta}^{RB}]$

8.2.8. 예제

Ex (8.2.10.) $X_1 \dots X_n \sim U[0,\theta]$, $\theta > 0$

$\hat{\eta}^{RB} \leftarrow \hat{\eta}^{MME}$)

$$\prod_{i=1}^{n} f(x_i;\theta) = \left(\frac{1}{\theta}\right)^n I(0 \le X_{(1)})\, I(X_{(n)} \le \theta).$$

$\Rightarrow X_{(n)}$: s.s. for θ.

• $E[X_1] = \frac{\theta}{2} \rightarrow \hat{\theta}^{MME} = 2\bar{X}$.

$MSE(\hat{\theta}^{MME}, \theta) = E((2\bar{X} - \theta)^2) = \frac{\theta^2}{3n}$. $\left\langle \begin{array}{l} E(\bar{X}) = \frac{\theta}{2} \rightarrow bias = 0. \\ Var(\bar{X}) = \frac{\theta^2}{12 \cdot n} \end{array}\right.$

• $E[2\bar{X} \mid X_{(n)} = y] = \frac{2}{n} E[X_1 + \dots + X_n \mid X_{(n)} = y]$

$= \frac{2}{n} E[X_{(1)} + \dots + X_{(n)} \mid X_{(n)} = y]$

$f(x_{(1)} \dots x_{(n)}) = n!\, f(x_1 \dots x_n) = n! \prod f(x_i)$.

• $f(x_{(1)} \dots x_{(n-1)} \mid X_{(n)} = y) = \dfrac{n! \left(\frac{1}{\theta}\right)^{n-1} \cdot \frac{1}{\theta}\, I(0 \le X_{(1)} \le \dots \le X_{(n-1)} \le y)}{n \left(\frac{y}{\theta}\right)^{n-1} \frac{1}{\theta}}$

$= (n-1)! \left(\frac{1}{y}\right)^{n-1} I(0 \le X_{(1)} \le \dots \le X_{(n-1)} \le y)$.

$\Rightarrow$ observe that $X_{(1)} \dots X_{(n-1)} \mid X_{(n)} = y \overset{d}{=} U_{(1)} \dots U_{(n-1)}$, $U_i \sim U(0,y)$.

$\Rightarrow E[X_{(1)} + \dots + X_{(n-1)} \mid X_{(n)} = y] = E[U_{(1)} + \dots + U_{(n-1)}]$

$= E[U_1 + \dots + U_{n-1}] = (n-1)\frac{y}{2}$.

$\Rightarrow \frac{2}{n} E[X_{(1)} + \dots + X_{(n)} \mid X_{(n)} = y] = \frac{2}{n}\left\{\frac{n-1}{2}y + y\right\} = \frac{n+1}{n} X_{(n)} \rightarrow \hat{\eta}^{RB}$

+) check: $Var(\hat{\eta}^{RB}) = Var\left(\frac{n+1}{n} X_{(n)}\right)$

$= \frac{\theta^2}{n(n+2)} \le \frac{\theta^2}{3n} = Var(\hat{\eta}^{MME})$.

※ $X_{(n)}$ 의 pdf: $\frac{n}{\theta^n} y^{n-1} \Rightarrow \frac{X_{(n)}}{\theta} \overset{d}{=} W \sim Beta(n,1)$.

$\left\langle E(W) = \frac{n}{n+1}. \quad Var(W) = \frac{n}{(n+1)^2(n+2)}. \right\rangle$.

8.3. 최소분산 불편추정

8.3.1. 정의

Def $X_1 \dots X_n \sim f(x;\theta)$.

(a) $Y = u(X_1 \dots X_n)$ is "complete" if $\forall \theta \in \Omega$, $E[g(Y)] = 0 \Leftrightarrow g(Y) = 0$ w.p.1. ($P_\theta(g(Y)=0)=1$)

(b) 〃 "complete s.s." if Y is s.s. for θ & complete.

8.3.2. 완비충분통계량과 UMVUE.

Thm 8.3.1. $X_1 \dots X_n \sim f(x;\theta)$, $Y = u(X_1 \dots X_n)$: CSS for θ. $\eta = \eta(\theta)$.

(a) $\hat{\eta}_0^{UE}$ (임의의 UE of $\eta(\theta)$)

$\Rightarrow$ $\hat{\eta}^{RB} = E[\hat{\eta}_0^{UE} \mid Y]$ 는 $\eta(\theta)$의 UMVUE.

(b) $\delta(Y)$ 가 $\eta(\theta)$의 UE $\Rightarrow$ $\delta(Y)$는 $\eta(\theta)$의 UMVUE.

pf) . 생략.

8.3.3. 완비충분통계량의 예

Ex 8.3.1. $X_1 \dots X_n \sim$ Bernoulli (θ). $0<\theta<1$.

sol) $Y = \sum X_i$: θ의 SS. $Y \sim Bin(n, \theta)$. $\rightarrow$ Y가 c.s.s. 임을 보이자.

$E_\theta[g(Y)] = 0$

$\Leftrightarrow \sum_{y=0}^{n} g(y) \binom{n}{y} p^y (1-p)^{\cancel{n-y}} = 0.$

$\Leftrightarrow \sum_y g(y) \binom{n}{y} \underbrace{\left(\frac{p}{1-p}\right)^y}_{= x^y} = 0.$

$\Leftrightarrow g(y) = 0$ w.p.1. ($\because$ 다항식 전개의 유일성)

$\Rightarrow E\left(\frac{Y}{n}\right) = \theta$ 이므로 $\frac{Y}{n} = \bar{X}$는 θ의 UMVUE.

Ex 8.3.2. $X_1 \dots X_n \sim$ Poisson (θ). $\theta > 0$

sol) $Y = \sum X_i$: θ의 SS. $Y \sim$ Poisson $(n\theta)$

$E_\theta[g(Y)] = 0$

$\Leftrightarrow \sum g(y) \frac{\cancel{e^{-n\theta}} (n\theta)^y}{y!} = 0.$

$\Leftrightarrow \sum^{\infty} \frac{g(y)}{y!} \cdot x^y = 0.$

$\Leftrightarrow g(y) = 0$ w.p.1. ($\because$ 멱급수 전개의 유일성).

$\Rightarrow E\left[\frac{Y}{n}\right] = \theta$ 이므로 $\frac{Y}{n} = \bar{X}$는 θ의 UMVUE

EX 8.3.3. $X_1 \dots X_n \sim Exp(\theta)$, $\theta > 0$.

sol) $Y = \sum X_i$: θ의 s.s. $Y \sim Gamma(n, \theta)$.

$E(g(Y)) = 0$

$\Leftrightarrow \int_0^\infty g(y) \frac{1}{\Gamma(n)\theta^n} y^{n-1} e^{-y/\theta} dy = 0$

$\Leftrightarrow \int_0^\infty g(y) y^{n-1} e^{-\lambda y} dy = 0$.

$\Leftrightarrow g(y) y^{n-1} = 0$. ($\because$ 라플라스 변환의 유일성) $\Leftrightarrow g(Y) = 0$ w.p.1.

$\Rightarrow E\left(\frac{Y}{n}\right) = \theta$ 이므로 $\frac{Y}{n} = \bar{X}$는 θ의 UMVUE.

8.3.4. 다중모수 지수족에서 완비충분통계량.

Thm 8.3.2. $f(x; \eta) = \exp\{\sum_{j=1}^k \eta_j T_j(x) - A(\eta) + S(x)\}$
(i) 분포의 Supp.
(ii) open ball 포함.
(iii) $c^T T(x) \neq c$ for $c_1 \dots c_k \neq 0$.

$\Rightarrow \sum T(X_i) = \left(\sum T_1(X_i), \dots, \sum T_k(X_i)\right)$: css for η.

8.3.5. UMVUE 예제들

EX 8.3.4. $X_1 \dots X_n \sim N(\mu, \sigma^2)$

sol) 분해정리 $\Rightarrow (\sum X_i, \sum X_i^2)$: c.s.s.

$\Rightarrow \left(\bar{X}, \frac{1}{n-1}\sum(X_i - \bar{X})^2\right) = (\bar{X}, S^2)$: c.s.s.

$E(\bar{X}) = \mu$, $E(S^2) = \sigma^2$ 이므로 $\bar{X}, S^2$는 μ, σ^2의 UMVUE.

EX 8.3.5. $X_1 \dots X_n \sim N(\mu, \mu^2)$.

sol) $f(x; \mu) = \frac{1}{\sqrt{2\pi}\mu} \exp\left\{-\frac{1}{2}\frac{(x-\mu)^2}{\mu^2}\right\} = \exp\left\{\frac{1}{\mu}x - \frac{1}{2\mu^2}x^2 - \frac{1}{2} - \frac{1}{2}\log(2\pi\mu^2)\right\}$.

지수족, but Thm 8.3.2. 적용 불가.

$Y = (\underbrace{\sum X_i}_{Y_1}, \underbrace{\sum X_i^2}_{Y_2}) \rightarrow E(Y_2/n) = EX_1^2 = Var(X_1) + (E(X_1))^2 = 2\mu^2$

$E((Y_1/n)^2) = E(\bar{X}^2) = Var(\bar{X}) + (E(\bar{X}))^2 = \left(\frac{1}{n}+1\right)\mu^2$

$g(Y) = \frac{1}{2n} Y_2 - \frac{1}{n(n+1)} Y_1^2$ 로 설정하면 $E(g(Y)) = 0$.

$\Rightarrow Y$는 완비통계량이 아니다.

$\bar{X}$는 μ의 UMVUE가 아니다.*
(교재X)

EX 8.3.6. $X_1 \dots X_n \sim Unif\ [0, \theta]$

sol) 분해정리 $\Rightarrow Y = X_{(n)}$: s.s. of θ.

$f_Y(y) = \frac{n}{\theta^n} y^{n-1} I(0 \le y \le \theta)$.

$E[g(Y)] = 0$

$\Leftrightarrow \int_0^\theta g(y) \frac{n}{\theta^n} y^{n-1} dy = 0$

$\Leftrightarrow g(y) y^{n-1} = 0$, $g(Y) = 0$ w.p.1. $\Rightarrow$ Y는 θ의 c.s.s.

$E[\frac{n+1}{n} Y] = \theta$ 이므로 $\frac{n+1}{n} X_{(n)}$: UMVUE for θ.

Ex 8.3.7. $X_1 \dots X_n \sim Bernoulli(\theta)$. $\rightarrow$ 모분산 $\eta = \theta(1-\theta)$의 UMVUE?

sol) $Y = \Sigma X_i$: css for θ.

(1) UE 찾기 : $E(X_1(1-X_2)) = E(X_1) \cdot E(1-X_2) = \theta(1-\theta)$.

$\Rightarrow \hat{\eta}_0^{UE} = X_1(1-X_2)$는 η의 U.E.

(2). R-B : $\hat{\eta}^{RB}(Y) = E[X_1(1-X_2) | Y]$

$= P(X_1 = 1, X_2 = 0 | X_1 + \dots + X_n = y)$.

$= \dfrac{P(X_1 = 1, X_2 = 0, X_3 + \dots + X_n = y-1)}{P(X_1 + \dots + X_n = y)}$

$= \dfrac{\theta(1-\theta) \cdot \binom{n-2}{y-1} \theta^{y-1} (1-\theta)^{n-y-1}}{\binom{n}{y} \theta^y (1-\theta)^{n-y}}$

$= y(n-y) / n(n-1)$. $\rightarrow$ UMVUE.

(3) $\hat{\theta} = \frac{Y}{n}$ 이라 하면 $\frac{Y(n-Y)}{n(n-1)} = \hat{\theta} \cdot \frac{n}{n-1}(1-\hat{\theta})$.

Ex 8.3.8. $X_1 \dots X_n \sim Poi(\theta)$. $\theta > 0$. $\rightarrow$ $\eta = e^{-2\theta}$의 UMVUE.

sol) $Y = \Sigma X_i$: c.s.s. for θ, $Y \sim Poi(n\theta)$

$E(f(Y)) = e^{-2\theta}$ 인 $f(Y)$를 찾으면 $f(Y)$: UMVUE of η.

$E(f(Y)) = \Sigma f(y) \dfrac{e^{-n\theta}(n\theta)^y}{y!} = e^{-2\theta}$

$\Leftrightarrow \Sigma \dfrac{f(y)}{y!} \lambda^y = e^{(n-2) \cdot \frac{\lambda}{n}}$ $\langle n\theta = \lambda \rangle$

$= \Sigma \frac{1}{y!} ((1 - \frac{2}{n})\lambda)^y$

$\Leftrightarrow f(Y) = (1 - \frac{2}{n})^y$.

$\Rightarrow f(Y) : \hat{\eta}^{UMVUE} = (1 - \frac{2}{n})^{n\bar{X}}$ 가 η의 UMVUE.

$(\simeq e^{-2\bar{X}} = \hat{\eta}^{MLE})$.

EX 8.3.9. $X_1 \dots X_n \sim N(\theta, 1)$, $\theta \in \mathbb{R}$, $\eta = P_\theta(X_1 > a)$. ▷ a에서의 신뢰도.

sol) 지수족 $\Rightarrow \bar{X}$: css of θ,

UE로 $\hat{\eta}^{UE} = I(X_1 > a)$.

$\rightarrow$ RB : $\hat{\eta}^{RB}(\bar{X}) = E[I(X_1>a) \mid \bar{X}]$

$= P(X_1 > a \mid \bar{X})$.

$P(X_1 > a \mid \bar{X} = y) = P(X_1 - \bar{X} > a - y \mid \bar{X} = y)$. 보조통계량

$= P(X_1 - \bar{X} > a - y)$ $(\because X_1 - \bar{X} \perp \bar{X})$

$= 1 - \Phi\left(\sqrt{\frac{n}{n-1}}(a-y)\right)$ $(\because X_1 - \bar{X} \sim N(0, \frac{n-1}{n}))$

$\Rightarrow \hat{\eta}^{RB}(\bar{X}) = 1 - \Phi\left(\sqrt{\frac{n}{n-1}}(a - \bar{X})\right) = \hat{\eta}^{UMVUE}$.

8.3.6. 보조통계량 (Basu's thm)

Thm 8.3.3. $Z = v(X_1 \dots X_n)$ 의 분포가 모든 $\theta \in \Omega$에 대해 X $\rightarrow$ Z : "보조통계량"

$\Rightarrow Y = u(X_1 \dots X_n)$: css for θ 와 Z는 독립.

pf) 생략.

8.3.7. 예제들

㊉ EX 8.3.10. $X_1 \dots X_n \sim Exp(\theta)$. $\theta > 0$ $\rightarrow$ $\eta = P_\theta(X_1 > a)$ 의 UMVUE?

sol) $Y = \sum X_i$: c.s.s. for $\theta > 0$, $Y \sim Gamma(n, \theta)$.

UE : $\hat{\eta}_0^{UE} = I(X_1 > a)$.

$\rightarrow$ RB : $\hat{\eta}^{RB}(Y) = E[I(X_1 > a) \mid Y] = P(X_1 > a \mid Y)$.

계산) $P(X_1 > a \mid Y = y) = P(X_1 > a \mid \sum X_i = y)$ $\quad Z_i \stackrel{d}{=} \frac{X_i}{\theta}$, $Z \sim Exp(1)$

$= P\left(\frac{X_1}{\sum X_i} > \frac{a}{y} \,\middle|\, \sum X_i = y\right)$ $\quad \frac{X_1}{\sum X_i} = \frac{Z_1}{\sum Z_i}$ $\rightarrow$ not depend on θ.

$= P\left(\frac{X_1}{\sum X_i} > \frac{a}{y}\right)$ $\quad (\sim Beta(1, n-1))$

$= \int_{\frac{a}{y}}^{1} (n-1)(1-t)^{n-2}\,dt = \left(1 - \frac{a}{y}\right)^{n-1} I(y \geq a)$.

$\Rightarrow \hat{\eta}^{RB}(\bar{X}) = \left(1 - \frac{a}{n\bar{X}}\right)^{n-1} I\left(\bar{X} \geq \frac{a}{n}\right)$. $\rightarrow$ UMVUE of η.

㊉ EX 추가 $X_1 \dots X_n \sim f$, $f(x; a, b) = \frac{1}{b}\exp\left\{-\frac{x-a}{b}\right\}$. $x \geq a$, $b > 0$.

(i) b가 주어진 경우 a의 UMVUE. (WLOG, b=1).

sol) (i) 분해정리) $\prod f(x_i; a, b) = \prod \exp(-x_i - a)\, \underbrace{I(x_i \geq a)}_{I(X_{(1)} \geq a)}$. $\rightarrow X_{(1)}$: s.s. for a.

(2) $Z_i \stackrel{d}{=} X_i - a \sim Exp(1) \Rightarrow X_{(1)}$: css for a.

$X_{(1)} - a \stackrel{d}{=} Z_{(1)} \sim Exp(\frac{1}{n})$ 이므로 $f_{X_{(1)}}(x;a) = n e^{-n(x-a)} I(x \geq a)$.

(3) 완비성) $E_a(g(X_{(1)})) = 0$

$\Leftrightarrow \int_a^\infty g(x)\, n e^{-n(x-a)} dx = 0$

$\Leftrightarrow \int_a^\infty g(x)\, n e^{-nx} dx = 0$

미분 $\Leftrightarrow -g(a)\, n e^{-na} = 0. \Leftrightarrow g(a) = 0 \ \forall a.$ $X_{(1)}$: css for a.

(4) UE인 $\delta(X_{(1)})$ 찾기) $E(\delta(X_{(1)})) = a$.

$E(X_{(1)}) = a + \frac{1}{n}$ 이므로 $\delta(X_{(1)}) = \underline{X_{(1)} - \frac{1}{n}}$ 으로 take

$\Rightarrow \hat{a}^{UMVUE}$.

(ii) a, b 모두 unknown 인 경우

sol) (1) $(X_{(1)}, \Sigma X_i)$: s.s.

$\Leftrightarrow (X_{(1)}, \Sigma(X_i - X_{(1)}))$

$\Leftrightarrow (X_{(1)} - a, \Sigma(X_i - X_{(1)}))$

(2) $\begin{cases} U = X_{(1)} - a \sim Exp(\frac{b}{n}). \\ V = \Sigma(X_i - X_{(1)}) \sim Gamma(n-1, b) \end{cases}$, $U \perp V$.

(3) 완비성) $\int g(u,v) f_U(u) f_V(v)\, du\, dv = 0 \Rightarrow$ 라플라스변환의 유일성에 의해 U, V는 c.s.s.

(4) $\hat{a}^{UMVUE}$) Let $\tilde{a} = \alpha X_{(1)} + \beta \bar{X}$.

$\hookrightarrow E(X_{(1)}) = a + \frac{b}{n}$, $E(\bar{X}) = a + b$.

$E(\tilde{a}) = \alpha(a + \frac{b}{n}) + \beta(a+b) = (\alpha+\beta) a + (\frac{\alpha}{n} + \beta) b. \ \forall a, b.$

$\forall a, b$ 에 대해 충족하려면 $\begin{cases} \alpha + \beta = 1, \\ \frac{\alpha}{n} + \beta = 0 \end{cases} \Rightarrow \alpha = \frac{n}{n-1}, \ \beta = -\frac{1}{n-1}.$

$\therefore \hat{a}^{UMVUE} = \frac{n}{n-1} X_{(1)} - \frac{1}{n-1} \bar{X}.$

(5) $\hat{b}^{UMVUE}$) Let $\tilde{b} = \gamma X_{(1)} + \delta \bar{X}$.

$E(\tilde{b}) = \gamma(a + \frac{b}{n}) + \delta(a+b) = (\gamma+\delta) a + (\frac{\gamma}{n} + \delta) b.$

$\rightarrow \begin{cases} \gamma + \delta = 0. \\ \frac{\gamma}{n} + \delta = 1 \end{cases} \Rightarrow \gamma = -\frac{n}{n-1}, \ \delta = \frac{n}{n-1}$

$\therefore \hat{b}^{UMVUE} = \frac{n}{n-1}(\bar{X} - X_{(1)}).$

⊛

Ex 추가2 $X_1 \dots X_n \sim Poi(\theta)$. $\theta > 0$. $\rightarrow$ $\eta = \theta^r \ (r \ge 1)$ 의 UMVUE?

sol) (1) 지수족 $\Rightarrow$ $S = \sum X_i$: css for θ, $S \sim Poi(n\theta)$.

(2)
$$E_\theta[S(S-1)(S-2)\cdots(S-r+1)] = \sum_{k=0}^{\infty} k(k-1)\cdots(k-r+1)\,P(S=k)$$
$$= \sum_{k=r}^{\infty} \frac{k!}{(k-r)!} \times \frac{e^{-n\theta}(n\theta)^k}{k!}$$
$$\overset{j=k-r}{=} \sum_{j=0}^{\infty} e^{-n\theta}\frac{(n\theta)^{j+r}}{j!}$$
$$= (n\theta)^r e^{-n\theta} \sum \frac{(n\theta)^j}{j!} = (n\theta)^r.$$

(3) $E\Big[\underbrace{\frac{1}{n^r} S(S-1)\cdots(S-r+1)}_{\delta(S)}\Big] = \underbrace{\theta^r}_{(\text{unbiased})}$ 이므로 $\delta(S)$ 는 η^{UMVUE}.

§8.4. 추정량의 점근적 비교

§8.4.1. 정의

Def (점근상대효율성) $\theta \leftarrow \hat\theta_n^1, \hat\theta_n^2$. s.t. $\sqrt{n}(\hat\theta_n^i - \theta) \xrightarrow{d} N(0, \sigma_i^2(\theta))$.

$\hat\theta_n^1$ 의 $\hat\theta_n^2$ 에 대한 ARE
$$ARE(\{\hat\theta_n^1\}, \{\hat\theta_n^2\}) = \frac{1/\sigma_1^2(\theta)}{1/\sigma_2^2(\theta)}.$$

§8.4.2. 예제들.

Ex 8.4.1. $X_1 \dots X_n \sim L(\theta, 1)$. $\theta \in \mathbb{R}$. $\rightarrow$ $ARE(\{\hat\theta_n^{mean}\}, \{\hat\theta_n^{med}\})$?

$$f(x;\theta) = \frac{e^{(x-\theta)}}{(1+e^{(x-\theta)})^2} \quad x \in \mathbb{R}.$$

sol) $\hat\theta_n^{med} = X_{([\frac{n+1}{2}])}$, $\hat\theta_n^{mean} = \bar{X}$.

- $\sqrt{n}(\hat\theta_n^{med} - \theta) \xrightarrow{d} N\left(0, \frac{1}{4(f(0))^2} = 4\right)$.
- $\sqrt{n}(\hat\theta_n^{mean} - \theta) \xrightarrow{d} N\left(0, \frac{\pi^2}{3}\right)$.

+) $\sqrt{n}(\hat\theta^{MLE} - \theta) \xrightarrow{d} N\left(0, \frac{1}{I(\theta)} = 3\right)$.

Ex 8.4.3. 위치모수 모형에서 중심값의 추정 : $f(x-\theta)$, $f(x) = f(-x)$ 인 모형.

$\sqrt{n}(\hat\theta_n^{med} - \theta) \xrightarrow{d} N(0, 1/4f^2(0))$

$\sqrt{n}(\hat\theta_n^{mean} - \theta) \xrightarrow{d} N(0, \sigma^2)$, $\sigma^2 = \int x^2 f(x)\,dx$.

$\sqrt{n}(\hat\theta^{MLE} - \theta) \xrightarrow{d} N(0, I^{-1})$, $I = Var\{\frac{\partial}{\partial\theta}\log f(X_1-\theta)\}$ ⊛
$$= \int \left(\frac{f'(x)}{f(x)}\right)^2 f(x)\,dx.$$

오차의 분포가

	$\frac{1}{4f^2(0)}$	σ^2	I_f^{-1}
$N(\theta,1)$	$\pi/2$	4	1
$L(\theta,1)$	1	$\pi^2/3$	2
$DE(\theta,1)$	1	3	1.

⊛ Ex 8.4.4. $X_1 \dots X_n \sim \text{Beta}(\alpha, 1).\ \alpha>0 \rightarrow ARE(\{\hat\alpha_n^{MME}\}, \{\hat\alpha_n^{MLE}\})$.

sol) $f(x:\alpha) = \alpha x^{\alpha-1} I(0<x<1). = \exp\{(\alpha-1)\log x + \log\alpha\} I(\,''\,)$

- $m_1 = \frac{\alpha}{\alpha+1} \rightarrow \hat\alpha_n^{MME} = \frac{\bar X}{1-\bar X}.\quad g(m_1) = \frac{m_1}{1-m_1},\quad \sigma^2 = \frac{\alpha}{\alpha+2} - \left(\frac{\alpha}{\alpha+1}\right)^2.$

  $= \frac{\alpha}{(\alpha+1)^2(\alpha+2)}$

  $\sqrt n(\hat\alpha_n^{MME} - \alpha) \rightarrow N(0, V)$

  ⊛ $\hookrightarrow (\dot g(m_1))^2 \cdot \sigma^2 = \frac{\alpha(\alpha+1)^2}{\alpha+2}$

- $\sum \log X_i + \frac{n}{\alpha} = 0. \leftrightarrow \hat\alpha^{MLE} = -\frac{1}{\overline{\log X}}.$

  $\sqrt n(\hat\alpha_n^{MLE} - \alpha) \rightarrow N(0, I^{-1}(\alpha))$

  $\rightarrow I(\alpha) = E\left[-\frac{\partial^2}{\partial\alpha^2}\log f(X_1:\alpha)\right] = \frac{1}{\alpha^2},\ I^{-1} = \alpha^2$

  $\therefore ARE = \frac{\alpha^2}{\frac{\alpha(\alpha+1)^2}{\alpha+2}}$

8.4.3. 정보량 부등식 (Cramer-Rao Bound)

Thm 8.4.1. $f(x:\theta)$. (R0) ~ (R7) 성립. 임의의 추정량 $\hat\eta_n$.

$\Rightarrow \mathrm{Var}_\theta(\hat\eta_n) \ge \left(\frac{\partial}{\partial\theta}E_\theta(\hat\eta_n)\right)^T [nI(\theta)]^{-1} \left(\frac{\partial}{\partial\theta}E_\theta(\hat\eta_n)\right) \quad \forall\theta\in\Omega.$

pf) (i) θ이 1차원. (ii) θ이 k차원.

Thm 8.4.2 (UMVUE에 대한 정보량 부등식). $\sqrt n(\hat\theta - \theta)$

θ의 UMVUE $\hat\theta_n^{UE}$

$\Rightarrow c^T \mathrm{Var}_\theta(\hat\theta_n^{UE})\, c \ge c^T [nI(\theta)]^{-1} c.\quad \forall\theta\in\Omega,\ \forall c.$

(일차원: $\mathrm{Var}_\theta(\hat\theta_n^{UE}) \ge [nI(\theta)]^{-1}.$) $\rightarrow \mathrm{Var}_\theta(\sqrt n(\hat\theta^{UE} - \theta)) \ge 1/I(\theta).$

점근정규성 성립하는 θ의 분산 $\sigma^2(\theta) \ge \frac{1}{I(\theta)}.$

$\sqrt n(\hat\theta^{MLE} - \theta) \rightarrow N(0, I^{-1}(\theta))$ 이므로

$\hat\theta^{MLE}$가 최소의 극한분산...

8.4.4. 예제들

$f(x:\alpha) = x^\alpha \sim U(0,1).$

⊛ Ex 8.4.5. $X_1 \dots X_n \sim \text{Beta}(\alpha, 1). \rightarrow$ UMVUE, 정보량 부등식.

sol) $f(x:\alpha) = \alpha x^{\alpha-1} I(0<x<1) = \exp\{(\alpha-1)\log x + \log\alpha\} I(\cdot)$

(1) 지수족 $\Rightarrow Y = -\sum\log X_i.$: CSS for α.

(2) $X_i^\alpha \sim U(0,1). \rightarrow -\alpha\log X_i \sim Exp(1).$

$-\alpha\sum\log X_i \sim Gamma(n,1).$

$\rightarrow Y \sim Gamma(n, \frac{1}{\alpha}),\quad f_Y(y) = \frac{1}{\Gamma(n)}\alpha^n y^{n-1} e^{-\alpha y}.$

(3) $\begin{cases} E\left[\dfrac{1}{Y}\right] = \dfrac{\alpha}{n-1}. & n \geq 2 \\ Var\left(\dfrac{1}{Y}\right) = \dfrac{\alpha^2}{(n-1)^2(n-2)} & n \geq 3 \end{cases} \implies \hat{\alpha}^{UMVUE} = (n-1) \cdot \dfrac{1}{Y}.$

$Var(\hat{\alpha}^{UMVUE}) = \dfrac{\alpha^2}{n-2}.$

(4) 정보량 부등식: $Var(\hat{\alpha}^{UMVUE}) \geq [nI(\alpha)]^{-1} = \alpha^2/n.$

(점근적으로 $\lim Var(\hat{\alpha}^{UMVUE}) / [nI(\theta)]^{-1} = 1.$)

(*) $\hat{\alpha}_n^{MLE} = \dfrac{n}{Y}, \to Var(\hat{\alpha}_n^{MLE}) = \dfrac{\alpha^2}{n}.$

$\begin{cases} \sqrt{n}(\hat{\alpha}^{UMVUE} - \alpha) \xrightarrow{d} N(0, \alpha^2) \\ \sqrt{n}(\hat{\alpha}^{MLE} - \alpha) \xrightarrow{d} N(0, \alpha^2). \end{cases} \to$ 점근분포는 동일.

☆ Ex 5.4.6.

$X_1 \dots X_n \sim Gamma(\alpha, \beta).$ $\alpha, \beta > 0.$ $\to ARE(\{\hat{\alpha}^{MME}\}, \{\hat{\alpha}^{MLE}\})$?

Sol) $f(x; \alpha, \beta) = \dfrac{1}{\Gamma(\alpha)\beta^\alpha} x^{\alpha-1} e^{-x/\beta} I(x>0).$

(1) 적률추정량 $\begin{bmatrix} \alpha\beta = \bar{X}. \\ \alpha\beta^2 = S_n^2 = \frac{1}{n}\sum(X_i - \bar{X})^2. \end{bmatrix} \implies \begin{bmatrix} \hat{\alpha}_n^{MME} = \dfrac{\bar{X}^2}{S_n^2} \\ \hat{\beta}_n^{MME} = \dfrac{S_n^2}{\bar{X}}. \end{bmatrix}$

(2). $\hat{\theta}^{MME} = \begin{bmatrix} \hat{\alpha}^{MME} \\ \hat{\beta}^{MME} \end{bmatrix}$의 점근분포: $\sqrt{n}\left(\hat{\theta}_n^{MME} - \theta\right) \xrightarrow{d} N(0, \Sigma(\theta)).$

↳ CLT, δ-method 이용 : $\Sigma(\theta) = \begin{pmatrix} 2\alpha(\alpha+1) & -2(\alpha+1)\beta \\ -2(\alpha+1)\beta & (2+\frac{3}{\alpha})\beta^2 \end{pmatrix}$

α MLE는 1차원으로 구한다.

(3) $I(\theta) = E_\theta[-\ddot{l}(\theta)] = \begin{bmatrix} \dot{\psi}(\alpha) & \frac{1}{\beta} \\ \frac{1}{\beta} & \alpha/\beta^2 \end{bmatrix}$

$I^{-1}(\theta) = \dfrac{1}{\alpha\dot{\psi}(\alpha) - 1} \begin{bmatrix} \alpha & -\beta \\ -\beta & \dot{\psi}(\alpha)\beta^2 \end{bmatrix}, \quad \sqrt{n}(\hat{\theta}_n^{MLE} - \theta) \xrightarrow{d} N(0, I^{-1}(\theta)).$

$\implies \begin{cases} ARE(\hat{\alpha}_n^{MME}, \hat{\alpha}_n^{MLE}) = \dfrac{1/2\alpha(\alpha+1)}{(\alpha\dot{\psi}(\alpha)-1)/\alpha} = \dfrac{1}{2(\alpha+1)(\alpha\dot{\psi}(\alpha)-1)} \\ ARE(\hat{\beta}_n^{MME}, \hat{\beta}_n^{MLE}) = \dfrac{1/(2+\frac{3}{\alpha})\beta^2}{(\alpha\dot{\psi}(\alpha)-1)/\dot{\psi}(\alpha)\beta^2} = \dfrac{\alpha\dot{\psi}(\alpha)}{(2\alpha+3)(\alpha\dot{\psi}(\alpha)-1)}. \end{cases}$

9장. 검정의 비교

9.1. 검정법 비교의 기준

9.1.1. 용어들

- 기각역 : $C(X)$

$\rightarrow$ Testing ftn $\phi(X) = \phi(X_1 \dots X_n) = \begin{cases} 1 & (X_1 \dots X_n) \in C(X) \\ r & (X_1 \dots X_n) \in r \cdot C(X). \\ 0 & (X_1 \dots X_n) \notin C(X) \end{cases}$

- 제 1종오류확률 $E_{\theta_0}(\phi(X_1 \dots X_n))$ v.s. 제 2종오류확률 $E_{\theta_1}(1-\phi(X_1 \dots X_n))$.
- 최대오류확률 : $\max\{E_{\theta_0}(\phi(X)),\ E_{\theta_1}(1-\phi(X))\}$
- 베이지안 평균오류확률 : (Average Risk). $\pi_0 E_{\theta_0}(\phi(X)) + \pi_1 E_{\theta_1}(1-\phi(X))$ $\rightarrow$ 사전확률 $P(H_0)=\pi_0,\ P(H_1)=\pi_1$.

9.1.2. 단순가설의 베이즈 검정

(정리 9.1.1) $H_0: \theta=\theta_0$ v.s. $H_1: \theta=\theta_1$.

$$\phi^B = \text{argmin}\{\text{베이지안 평균오류확률}\} = \begin{cases} 1 & \dfrac{f(x:\theta_1)}{f(x:\theta_0)} > \dfrac{\pi_0}{\pi_1} \\ 0 & \quad '' \quad < \dfrac{\pi_0}{\pi_1} \end{cases}$$

pf) $$\text{A. Risk} = \pi_0 \int \phi(x) f(x:\theta_0)dx + \pi_1 \int (1-\phi(x)) f(x:\theta_1)dx$$
$$= \pi_1 + \int \phi(x)\{\underbrace{\pi_0 f(x:\theta_0) - \pi_1 f(x:\theta_1)}\}dx$$

(−) 일 때 $\rightarrow$ $\phi(x)=1$.
(+) 일 때 $\rightarrow$ $\phi(x)=0$.

9.1.3. Neyman-Pearson 패러다임 (전통적인 검정의 형식논리).

① (Simple) $E_{\theta_0}[\phi(X)] \le \alpha$를 만족시키는 ϕ들 중 $\max E_{\theta_1}[\phi(X)]$ 인 ϕ.
검정력.

$\Rightarrow$ Most Powerful Test.

② (Composite) $H_0: \theta \in \Omega_0$ vs. $H_1: \theta \in \Omega_1$.

$\sup_{\theta \in \Omega_0} E_\theta(\phi(X)) \le \alpha$ 중에서 $\max E_{\theta_1}[\phi(X)]\ \forall \theta_1 \in \Omega_1$ 인 ϕ.

$\Rightarrow$ Uniformly MP Test.

9.1.4. 최강력 검정. ϕ^{MP_α}.

(i) $E_{\theta_0}(\phi^{MP_\alpha}(X)) \le \alpha$.

(ii) $\forall \phi$: $E_{\theta_0}(\phi(X)) \le \alpha$, $E_{\theta_1}(\phi^{MP}(X)) \ge E_{\theta_1}(\phi^{MP}(X))$.

9.1.5. 단순가설의 최강력검정. $H_0: \theta=\theta_0$. $H_1: \theta=\theta_1$.

(정리 9.1.2.) (a) $\phi^*(X) = \begin{cases} 1 & \dfrac{f(x:\theta_1)}{f(x:\theta_0)} > k \\ r & \quad '' \quad = k \\ 0 & \quad '' \quad < k \end{cases}$ (b) $E_{\theta_0}(\phi^*(X)) = \alpha$.

가능도비 검정꼴

$\Rightarrow \phi^{MP_\alpha}$.

pf) N.T.S. $E_{\theta_1}\phi^*(X) \ge E_{\theta_1}\phi(X)$ $\quad {}^{\forall}\, 0\le \phi(X)\le 1$. ← 유의수준 α.

$$E_{\theta_1}\phi(X) - kE_{\theta_0}\phi(X) = \int \phi(x)\frac{f(x:\theta_1)}{f(x:\theta_0)}f(x:\theta_0)\,dx - k\int \phi(x) f(x:\theta_0)\,dx$$

$$= \int \phi(x)\left\{\frac{f(x:\theta_1)}{f(x:\theta_0)} - k\right\} f(x:\theta_0)\,dx + \int \phi(x) f(x:\theta_1)\,dx. \quad \text{Iff}(x:\theta_0)=0).$$

By (a)

$$\Rightarrow \left(E_{\theta_1}\phi^*(X) - kE_{\theta_0}\phi^*(X)\right) - \left(E_{\theta_1}\phi(X) - kE_{\theta_0}\phi(X)\right)$$

$$= E_{\theta_0}\left[(\phi^*(X)-\phi(X))\left\{\frac{f(x:\theta_1)}{f(x:\theta_0)} - k\right\}\right] + E_{\theta_1}\left[(1-\phi(X))\, I(f(x:\theta_0)=0)\right] \ge 0.$$

$$\Leftrightarrow E_{\theta_1}\phi^*(X) - E_{\theta_1}\phi(X) \ge k\left\{E_{\theta_0}\phi^*(X) - E_{\theta_0}\phi(X)\right\} = k\left\{\underset{(b)}{\alpha} - \underset{\le\alpha}{E_{\theta_0}\phi(X)}\right\} \ge 0.$$

9.1.6. (예제 9.1.2.) $X_1 \dots X_{100} \sim Poi(\theta)$, $H_0: \theta = 0.1$ vs. $H_1: \theta = 0.05$

$$f(x:\theta_1)/f(x:\theta_0) = e^{-n(\theta_1-\theta_2)} \cdot (\theta_1/\theta_2)^{x_1+\dots+x_{100}}.$$

$$\Rightarrow \phi^* = \begin{cases} 1 & x_1+\dots+x_{100} \le c-1 \\ \gamma & \quad '' \quad = c \\ 0 & \quad '' \quad \ge c+1 \end{cases} \quad \& \; E_{\theta_0}\phi^*(X) = 0.05 \Rightarrow c=5,\; \gamma = 21/38 \text{ 로 결정}.$$

9.2. 전역최강력 검정.

9.2.1. 정의.

Def $H_0: \theta \in \Omega_0$ vs. $H_1: \theta \in \Omega_1$ → ϕ^{UMP_α}

(i) (유의수준 α) $\max_{\theta\in\Omega_0} E_\theta[\phi^{UMP_\alpha}] \le \alpha$.

(ii) (대립가설 전역에서 MP) $E_{\theta_1}\phi^{UMP_\alpha}(X) \ge E_{\theta_1}\phi(X)$ ${}^{\forall}\theta_1 \in \Omega_1$, ${}^{\forall}\phi: \max_{\theta\in\Omega_0} E_\theta\phi(X) \le \alpha$. (조건(i))

9.2.2. 단일모수 지수족. 한쪽검정의 UMP

(정리 9.2.1.) $H_0: \theta \le \theta_0$ vs. $H_1: \theta > \theta_0$.

$f(x:\theta) = \exp\{g(\theta)T(x) - B(\theta) + S(x)\}$ $\quad g: \theta$의 증가함수.

$$\Rightarrow \phi^{UMP_\alpha}: \text{(a) (가능도비 검정 꼴)}. \quad \phi^*(X) = \begin{cases} 1 & \sum T(X_i) > c \\ \gamma & \quad '' \quad = c \\ 0 & \quad '' \quad < c. \end{cases}$$

(b) (검정의 크기) $E_{\theta_0}\phi^*(X) = \alpha$.

pf) ① 작은 가설 $H_0: \theta = \theta_0$ v.s. $H_1: \theta = \theta_1 > \theta_0$.

$$f(x:\theta_1)/f(x:\theta_0) = \exp\{(g(\theta_1)-g(\theta_0))(T(x_1)+\dots T(x_n)) - B(\theta_1)+B(\theta_0)\}.$$

→ $T(x_1)+\dots+T(x_n)$의 증가함수.

$$\phi^*(X) = \begin{cases} 1 & \sum T(X_i) > c \\ \gamma & = c \\ 0 & < c \end{cases}, \quad E_{\theta_0}[\phi^*(X)] = \alpha \text{ 인 } \gamma, c \text{ 찾는다}. \quad (MP_\alpha).$$

② ϕ^*는 θ_1에 의존X. ⇒ $H_0: \theta=\theta_0$ vs. $H_1: \theta>\theta_0$ 의 UMP_α.

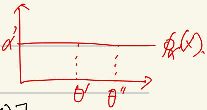
③ claim) $E_\theta \phi^{*}(X)$가 θ의 증가함수 ($\Rightarrow \max_{\theta\in\Theta_0} E_\theta\phi^*(X) = E_{\theta_0}\phi^*(X)=\alpha$).

임의의 θ', θ'' → $H_0:\theta=\theta'$ vs. $H_1:\theta=\theta''$

$MP_{\alpha'}$ test : $\phi^{**} = \begin{cases}1 & \Sigma T(x_i) > c' \\ \gamma' & = c' \\ 0 & < c'\end{cases}$, 유의수준 $\alpha' = E_{\theta'}[\phi^*(X)]$로 setting

⇒ $\phi^{**}=\phi^*$, (∵ MP의 유일성).

$\phi_{\alpha'} = \begin{cases}1 & \alpha' \\ 0 & 1-\alpha'\end{cases}$ 인 검정.



$E_{\theta''}[\phi^*(X)] \ge E_{\theta''}[\phi_{\alpha'}(X)] = \alpha' = E_{\theta'}[\phi^*(X)]$.

∴ $E_\theta\phi^*(X)$는 θ의 증가함수.

9.2.3. 정규분포 평균 단측 UMP. $N(\mu,1)$

(예제 9.2.1.) $H_0:\mu=\mu_0$ vs. $H_1:\mu>\mu_0$. → $H_1:\mu=\mu_1>\mu_0$에 대한 UMP

sol) $\phi^*(x)=\begin{cases}1 & f(x;\mu_1)/f(x;\mu_0)\ge c \\ 0 & \quad '' \quad < c\end{cases}$ ⇔ $\bar x \ge c$, $\bar x < c$. & $E_{\mu_0}[\phi^*(X)]=\alpha$.

↳ $\exp\{n(\mu_1-\mu_0)(\bar x - \frac{\mu_1+\mu_0}{2})\}$: $\bar x$에 대한 증가함수.

Under H_0, $\bar X\sim N(\mu_0,\frac1n)$,

$c=\mu_0+z_\alpha/\sqrt n$. → μ_1에 관계없이 성립.

(예제 9.2.2.) $H_0^-: \mu\le\mu_0$ vs. $H_1:\mu>\mu_0$. UMP.

sol) ① $E_\mu[\phi^*(X)] = P(\bar X-\mu_0 \ge z_\alpha/\sqrt n)$

H_0^-에서도 유의수준=α.

$= P(\sqrt n(\bar X-\mu)\ge\sqrt n(\mu_0-\mu)+z_\alpha)$

$= P(Z\ge \sqrt n(\mu_0-\mu)+z_\alpha)$ ∝ μ.

∴ $\max_{\mu\le\mu_0} E_\mu[\phi^*(X)] = E_{\mu_0}[\phi^*(X)] = \alpha$.

② H_0^-에서 유의수준 α인 검정은 H_0에 대해서도 유의수준 α.

$\{\phi: \max_{\mu\le\mu_0} E_\mu\phi(X)\le\alpha,\ 0\le\phi(X)\le1\} \subseteq \{\phi: E_{\mu_0}\phi(X)\le\alpha,\ 0\le\phi(X)\le1\}$.

여기서도 최강력 검정이다. ← φ*가 최강력 검정이다.

9.2.4. 지수분포$Exp(\theta)$, 평균, 단측, UMP. $H_0:\theta\le\theta_0$. vs. $H_1:\theta>\theta_0$.

(예제 9.2.4.) $f(x;\theta)=\exp\{-\frac{x}{\theta}-\log\theta\}$

sol) By (정리 9.2.2.), $\phi^*(X)=\begin{cases}1 & X_1+\cdots+X_n\ge c\\0 & \quad '' \quad <c\end{cases}$ & $E_{\theta_0}\phi^*(X)=\alpha$.

Under H_0, $\theta=\theta_0$: $\dfrac{\sum X_i}{\theta_0} \sim \text{Gamma}(n,1)$, $2\dfrac{\sum X_i}{\theta} \sim \text{Gamma}(n,2) = \chi^2(2n)$.

$$\Rightarrow c = \frac{1}{2}\theta\, \chi^2_\alpha(2n)$$

9.2.5. 정규분포 $N(\mu,1)$ 평균, 양측검정 UMP → 존재 X.

(예제 9.2.5.) $H_0: \mu=\mu_0$ vs. $H_1: \mu\neq\mu_0$.

pf) Suppose $\exists\, \phi^*$: UMP.

→ $H_0: \mu=\mu_0$ vs. $H_1: \mu=\mu_1 < \mu_0$ / $H_0: \mu=\mu_0$ vs. $H_1: \mu=\underline{\mu_2>\mu_0}$.

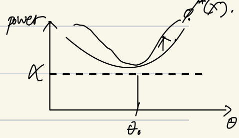
에 대하여 유의수준 α의 MP여야 한다.

i.e. $\exists\, c_1, c_2$ s.t. $\phi^*(x) = \begin{cases} 1 & \bar{X} \le c_1 \\ 0 & \bar{X} > c_1 \end{cases}$, $\phi^*(x) = \begin{cases} 1 & \bar{X} \ge c_2 \\ 0 & \bar{X} < c_2 \end{cases}$.

→ 이를 만족하는 c_1, c_2 존재X.

9.2.6. 정규분포 $N(\theta,1)$. 평균. 양측검정의 성질. — 최대가능도비 검정.

(예제 9.2.6.) $H_0: \theta=\theta_0$ vs. $H_1: \theta\neq\theta_0$, $\phi^* = \begin{cases} 1 & |\sqrt{n}(\bar{X}_n-\mu_0)| > z_{\alpha/2} \\ 0 & \quad '' \quad \le z_{\alpha/2} \end{cases}$ (ϕ^{CR})



⇒ 다음이 성립한다:

(a) $E_{\theta_0}[\phi^*(X)] = \alpha$, $\dfrac{d}{d\theta} E_\theta[\phi^*(X)]\Big|_{\theta=\theta_0} = 0$.

(b) $\forall\, \phi(x)$ with (a), $\forall\, \theta_1\neq\theta_0$, $E_{\theta_1}\phi^*(X) \ge E_{\theta_1}\phi(X)$.

pf) (a) $\beta(\theta) = E_\theta[\phi^*(X)]$.

$$= P_\theta\left(|\sqrt{n}(\bar{X}_n-\theta_0)| > z_{\alpha/2}\right).$$

$$= 1-\{\Phi(z_{\alpha/2}-\delta) - \Phi(-z_{\alpha/2}-\delta)\} \qquad \delta = \sqrt{n}(\theta-\theta_0).$$

$$\Rightarrow \beta(\theta_0) = \alpha. \qquad \frac{d}{d\theta}\beta(\theta)\Big|_{\theta=\theta_0} = 0.$$

(b) 라그랑주 승수법 이용.

$$\left(* \quad \frac{d}{d\theta_0}E_{\theta_0}[\phi(X)] = \frac{d}{d\theta_0}\int \phi(x) f(x:\theta_0)\,dx = \int \phi(x)\frac{f'(x:\theta_0)}{f(x:\theta_0)} f(x:\theta_0)\,dx = E_{\theta_0}\left[\phi(X)\frac{f'(x:\theta_0)}{f(x:\theta_0)}\right].\right)$$

$$E_{\theta_1}[\phi(X)] - k_1 E_{\theta_0}[\phi(X)] - k_2 E_{\theta_0}\left[\phi(X)\frac{f'}{f}\right]$$

$$= E_{\theta_0}\left[\phi(X)\frac{f_1}{f_0} - k_1\phi(X) - k_2\frac{\dot{f}_0}{f_0}\right] = E_{\theta_0}\left[\phi(X)\left\{\frac{f_1}{f_0} - k_1 - k_2\frac{\dot{f}_0}{f_0}\right\}\right] \cdots (*)$$

$$\underset{\phi}{\text{argmin}}\,((*)) = \phi^{**} = \begin{cases} 1 & \frac{f_1}{f_0} \ge k_1 + k_2\frac{\dot{f}_0}{f_0} \\ 0 & \quad '' \quad < k_1 + k_2\frac{\dot{f}_0}{f_0} \end{cases}$$

$$\Rightarrow E_{\theta_1}[\phi^{**}(X)] - E_{\theta_1}[\phi(X)] \ge k_1\{E_{\theta_0}[\phi^{**}(X)] - E_{\theta_0}[\phi(X)]\}$$

$$+ k_2\left\{E_{\theta_0}\left[\phi^{**}\frac{\dot{f}_0}{f_0}\right] - E_{\theta_0}\left[\phi\frac{\dot{f}_0}{f_0}\right]\right\}.$$

ϕ^{**}가 (a) 만족시키도록 k_1, k_2 take $\Rightarrow$ $E_{\theta_1}\phi^{**}(X) \geq E_{\theta_1}\phi(X)$.

$\Rightarrow \frac{f_1}{f_0} \geq k_1 + k_2 \frac{f_{\alpha}}{f_0}$ 조건 $\underbrace{\sum X_2 < C_1, \quad \sum X_2 > C_2}$ 로 표현된다.

$C_1 = -C_2, \quad C_2 = \frac{1}{\sqrt{n}} z_\alpha + \mu_0 \ldots$

9.2.7. 단일모수 지수족, 양측, ump 존재 X.

$$\begin{cases} f(x:\theta) = \exp\{g(\theta)T(x) - \beta(\theta) + S(x)\}. \quad g(\theta): \theta\text{에 대한 강증가.} \\ H_0: \theta = \theta_0 \quad vs. \quad H_1: \theta \neq \theta_0. \end{cases}$$

$$\phi^*(X) = \begin{cases} 1 & T(X) < C_1, \ T(X) > C_2 \\ \gamma & T(X) = C_1, \ T(X) = C_2 \\ 0 & C_1 < T(X) < C_2. \end{cases}$$

$\Rightarrow$ (a)(b) 만족하도록 C_1, C_2 정한다. $\rightarrow$ S.S.로 표현?

다른 조건) $\begin{cases} E_{\theta_0}[\phi(T)] = \alpha. \\ E_{\theta_0}[\phi(T)(T - E_\theta(T))] = 0. \end{cases}$

pf) $\frac{d}{d\theta} E_\theta[\phi^*(X)]\Big|_{\theta=\theta_0} = 0$ 유도:

$$\frac{d}{d\theta}\int \phi(x) f(x:\theta)\,dx = \int \phi(x) \frac{d}{d\theta} f(x:\theta)\,dx$$
$$= \int \phi(x)\{g'(\theta)T(x) - \beta'(\theta)\} f(x:\theta)\,dx$$
$$= g'(\theta) E_\theta[\phi(X)T(X)] - \beta'(\theta)E_\theta[\phi(X)]. \ (=0).$$

$E_\theta(T(X)) = \frac{\beta'(\theta)}{g'(\theta)}$ 이므로 대입.

$$E_{\theta_0}[\phi(X)T(X)] = E_{\theta_0}[\phi(X)] \cdot E_{\theta_0}[T(X)]. \Rightarrow \phi, T \text{ 간 공분산} = 0$$

$$\Leftrightarrow E_{\theta_0}[\phi(X)(T - E_{\theta_0}(T(X)))] = 0.$$

9.3. 비모수적 검정의 점근적 비교.

9.3.1. 검정력의 근사와 표본크기.

(정리 9.3.1.) $H_0: \theta = \theta_0$ vs. $H_1: \theta > \theta_0$ 검정통계량 T_n.

① 유의수준 α 기각역: $\sqrt{n}(T_n - \mu(\theta_0))/\sigma(\theta_0) \geq t_n \simeq z_\alpha$.

② $\sqrt{n}(T_n - \mu(\theta))/\sigma(\theta) \xrightarrow{d} N(0,1), \ \forall\theta.$

③ $\mu(\theta), \sigma(\theta)$: θ에 대해 미분가능 & 연속.

$\Rightarrow$ (a) (검정력). $\theta_{1n} \simeq \theta_0 + k/\sqrt{n}$. 에 대해

검정력 $\gamma_n(\theta_{1n}) = P_{\theta_{1n}}(\sqrt{n}(T_n - \mu(\theta_0))/\sigma(\theta_0) \geq t_n)$.

$$\simeq 1 - \Phi\left(-\sqrt{n}(T_n - \theta_0)\frac{\dot{\mu}(\theta_0)}{\sigma(\theta_0)} + z_\alpha\right).$$

(b) (표본크기의 근사) $\gamma_n(\theta_{1n}) \simeq \gamma$ 이기 위한 표본크기 $N(T_n: \gamma, \theta_{1n}) \simeq \left(\frac{\dot{\mu}(\theta_0)}{\sigma(\theta_0)}\right)^{-2}\left(\frac{z_\alpha + z_{1-\gamma}}{\theta_{1n} - \theta_0}\right)^2$

pf) (a) $\theta=\theta_1>\theta_0$ 의 검정력: $\gamma(\theta_1) = P_{\theta_1}\left(\frac{\sqrt{n}(T_n-\mu(\theta_0))}{\sigma(\theta_0)} > t_n\right).$

$$= P_{\theta_1}\left(\frac{\sqrt{n}(T_n-\mu(\theta_1))}{\sigma(\theta_1)} \cdot \frac{\sigma(\theta_1)}{\sigma(\theta_0)} + \frac{\sqrt{n}(\mu(\theta_1)-\mu(\theta_0))}{\sigma(\theta_0)} \geq t_n\right).$$

$P_{\theta_0}\left(\sqrt{n}(T_n-\mu(\theta_0))/\sigma(\theta_0) \geq t_n\right)=\alpha.$

$\to P(Z \geq t_n) \simeq \alpha.$

$\to t_n \simeq z_\alpha.$

$$\simeq P_{\theta_1}\left(Z \geq \frac{-\sqrt{n}(\mu(\theta_1)-\mu(\theta_0))}{\sigma(\theta_0)} + \frac{\sigma(\theta_0)}{\sigma(\theta_1)} z_\alpha\right).$$

$\simeq \dot{\mu}(\theta_0)(\theta-\theta_0).$ $\theta_{1n} = \theta_0 + k/\sqrt{n}.$ $\Rightarrow \sigma(\theta_0)/\sigma(\theta_1) \to 1.$

$\theta_1 \Rightarrow \theta_{1n}$

$\therefore \gamma(\theta_{1n}) = 1 - \Phi(\quad " \quad)$

(b) $1-\Phi(-A) = \gamma \iff A = z_{1-\gamma}.$

$\iff \sqrt{n}(\theta_{1n}-\theta_0)\dfrac{\dot{\mu}(\theta_0)}{\sigma(\theta_0)} - z_\alpha = z_{1-\gamma}.$

$\iff n = \left(\dfrac{z_\alpha + z_{1-\gamma}}{\theta_{1n}-\theta_0}\right)^2 \cdot \left[\dfrac{\dot{\mu}(\theta)}{\sigma(\theta)}\right]^{-2}.$

9.3.2. 점근상대효율성 (ARE).

Def T_{1n}, T_{2n} : (점근정규성) $\sqrt{n}\dfrac{(T_{1n}-\mu_1(\theta))}{\sigma_1(\theta)} \to N(0,1)$, $\sqrt{n}\dfrac{(T_{2n}-\mu_2(\theta))}{\sigma_2(\theta)} \to N(0,1).$

$$\Rightarrow ARE(\{T_{1n}\},\{T_{2n}\}) \triangleq \lim_{n\to\infty} \frac{N(T_{1n};\gamma,\theta_{1n})^{-1}}{N(T_{2n}:\gamma,\theta_{1n})^{-1}}$$

$$= \frac{[\dot{\mu}_1(\theta)/\sigma_1(\theta)]^2}{[\dot{\mu}_2(\theta)/\sigma_2(\theta)]^2}.$$

9.3.3. 위치모수 모형에서의 부호검정.

(예제 9.3.1.) $\begin{cases} X_1 \dots X_n \sim \frac{1}{\sigma}f\left(\frac{x-\theta}{\sigma}\right). \to \text{suppose } \sigma=1. \\ f(z) = f(-z). \end{cases}$

중앙값. $f(x-\theta)$.

$H_0: \theta = \theta_0$ vs. $H_1: \theta > \theta_0.$

$\Rightarrow$ 부호검정 $S_n = \sum I(X_i > \theta). \sim Bin(n, p(\theta) = P_\theta(X_1>\theta_0) = 1-F(\theta_0-\theta)).$ true.

$\Rightarrow H_0: p(\theta) = \frac{1}{2}$ vs. $H_1: p(\theta) > \frac{1}{2}$ 이 됨.

$$\Rightarrow \begin{cases} \phi_S(X) = \begin{cases} 1 & S_n \geq c+1 \\ \gamma & S_n = c \\ 0 & S_n < c-1 \end{cases} \\ E_{\theta_0}[\phi_S(X)] = P_{\theta_0}(S_n \geq c+1) + \gamma P(S_n=c) = \alpha. \end{cases}$$

+) $E_\theta \phi_S(X_1 \dots X_n) = E_\theta \phi_S(Z_1+\theta, \dots, Z_n+\theta)$, $Z_i \sim f.$

$S_n \stackrel{d}{=} \sum Z_i + n\theta$ 이므로 $E_\theta[\phi_S(X)]$는 θ에 대한 증가함수.

$$\max_{\theta\leq\theta_0} E_\theta \phi_S(X) = E_{\theta_0}\phi_S(X) = \alpha.$$

$\Rightarrow H_0: \theta \leq \theta_0$ 의 유의수준 α 검정.

9.3.4. 위치모수 모형. 부호검정 검정력 근사.

(예제 9.3.2.) S_n의 극한분포 : $\dfrac{S_n - np(\theta)}{\sqrt{n\sigma^2(\theta)}} \xrightarrow{d} N(0,1)$. $\sigma^2(\theta) = p(\theta)(1-p(\theta))$.

기각역 : $S_n \geq C_n \simeq np(\theta_0) + \sqrt{n}\sigma(\theta_0) z_\alpha$. $p(\theta_0) = \frac{1}{2} = \sigma(\theta_0)$.

$\frac{1}{n}S_n \triangleq T_n$. $\Rightarrow$ 검정력 $\gamma_n(\theta_1) = 1 - \Phi\left\{-\sqrt{n}\dfrac{p(\theta_1) - p(\theta_0)}{\sigma(\theta_1)} + \dfrac{\sigma(\theta_0)}{\sigma(\theta_1)} z_\alpha\right\}$. ~~d~~ $\gamma_n(\theta_1) = 1$.

$E(S_n) = np(\theta)$
$Var(S_n) = np(\theta)(1-p(\theta))$
$\Rightarrow \mu(\theta) = p(\theta) = 1 - F(\theta_0 - \theta)$.
$\sigma^2(\theta) = p(\theta)(1-p(\theta))$

$\theta_n \simeq \theta_0 + k/\sqrt{n}$에 대해 $\gamma_n(\theta_n) \simeq 1 - \Phi\left\{-\sqrt{n}\dfrac{(\theta_n - \theta_0)\dot{p}(\theta_0)}{\sigma(\theta_0)} + z_\alpha\right\}$.

$\dot{p}(\theta_0) = f(\theta_0 - \theta)$.

$\Rightarrow \gamma_n(\theta_n) \simeq \gamma$ 이기 위한 표본크기 $n \simeq (2f(0))^{-2}\left(\dfrac{z_\alpha + z_{1-\gamma}}{\theta_n - \theta}\right)^2$.

9.3.5. 부호검정 ARE

(예제 9.3.3.) $X_1 \dots X_n \sim \frac{1}{\sigma}f\left(\frac{x-\theta}{\sigma}\right)$, $H_0 : \theta \leq \theta_0$ vs. $H_1 : \theta > \theta_0$. ☆

(i) 부호검정. $S_n = \Sigma I(X_i > \theta_0)$. $\sim B(n, \underbrace{p(\theta)}_{\to 1-F\left(\frac{\theta_0-\theta}{\sigma}\right)})$.

$\dfrac{\sqrt{n}\left(\frac{1}{n}S_n - \mu(\theta)\right)}{\sigma(\theta)} \xrightarrow{d} N(0,1)$. $\therefore \mu(\theta) = p(\theta) = 1 - F\left(\frac{\theta_0-\theta}{\sigma}\right)$.
$\sigma^2(\theta) = p(\theta)(1-p(\theta))$.

(ii) t-검정. $t_n = \dfrac{\bar{X} - \theta_0}{S/\sqrt{n}}$.

Let $T_n = \dfrac{\bar{X} - \theta_0}{S}$ $\rightarrow \mu(\theta) = \dfrac{\theta - \theta_0}{\sqrt{Var(X)}}$. $\sigma(\theta) = 1$

$\sqrt{n}\left[\dfrac{\bar{X} - \theta_0}{S} - \dfrac{\theta - \theta_0}{\sqrt{Var(X_i)}}\right] \xrightarrow{d} N(0,1)$.
$S \xrightarrow{P} \sqrt{Var(X_i)}$.

$\therefore \dot{\mu}(\theta) = \dfrac{1}{\sqrt{Var(X)}}$.

* $Var(X_i) = \int (x-\theta)^2 \frac{1}{\sigma}f\left(\frac{x-\theta}{\sigma}\right) dx$; $\frac{x-\theta}{\sigma} = z$.
$= \int \sigma^2 z^2 f(z)\, dz$

$\Rightarrow$ ARE 계산 : $\left(\dfrac{2f(0)}{\sigma}\right)^2 / \left(\dfrac{1}{\sqrt{Var(X_i)}}\right)^2 = 4f(0)^2 \int z^2 f(z)\, dz$

$N(\mu,\sigma^2)$	$\frac{2}{\pi}$
$L(\mu,\sigma)$	$\frac{\pi^2}{12}$
$DE(\mu,\sigma)$	2.

9.3.6. 위치모형. 부호순위검정.

(예제 9.3.4.) $X_1 \dots X_n \sim f(x-\theta)$. $H_0 : \theta = \theta_0$ vs $H_1 : \theta > \theta_0$.

$W_n = \Sigma\, sgn(X_i - \theta_0) R(|X_i - \theta_0|)$. $\to$ $|X_1 - \theta_0| \dots |X_n - \theta_0|$ 에서 $|X_i - \theta_0|$의 순위.

Under H_0, $W_n \overset{d}{=} \sum_{j=1}^n Z_j \cdot j$ $Z_j \overset{iid}{\sim} \begin{cases} 1 & \frac{1}{2} \\ -1 & \frac{1}{2} \end{cases}$

$\Rightarrow E(W_n) = 0$. $Var(W_n) = \sum_{j=1}^n j^2 \underbrace{Var(Z_j)}_{=1} = \dfrac{n(n+1)(2n+1)}{6}$.

$\Rightarrow \dfrac{W_n - E(W_n)}{Var(W_n)} \xrightarrow{d} N(0,1)$.

$\to [\mathrm{sgn}(X_i-\theta_0)+1]\cdot\frac{1}{2}$

(정리 9.3.3.) Let $W_n^+ = \sum_{i=1}^{n} I(X_i > \theta_0)\, R(|X_i-\theta_0|)$.

$\Rightarrow$ (a) $W_n = 2W_n^+ - n(n+1)/2$

(b) Under $H_0: \theta=\theta_0$, $W_n^+ \overset{d}{=} \sum_j j B_j$. $B_j \sim \mathrm{Bernoulli}(\frac{1}{2})$.

(증명X) $\Rightarrow$ $\dfrac{W_n^+ - \frac{n(n+1)}{4}}{\sqrt{n(n+1)(2n+1)/24}} \xrightarrow{d} N(0,1)$.

(정리 9.3.4.) $W_n^+ = \sum_{1\le i\le j\le n} I(X_i+X_j \ge 2\theta_0)$

pf) $Y_i = X_i - \theta$, $R(|Y_i|) = 1 + \sum I(|Y_i| < |Y_j|)$.

$W_n^+ = \sum I(Y_i > 0)\, R(|Y_i|)$.

9.3.7. 부호순위 검정의 점근정규성

(정리 9.3.6.) (a) $\dfrac{W_n^+ - E_\theta(W_n^+)}{Var_\theta(W_n^+)} \xrightarrow{d} N(0,1)$.

(b) $\mu(\theta) = \frac{1}{2} P_\theta(X_1+X_2 > 2\theta_0)$

$\sigma^2(\theta) = Cov_\theta(I(X_1+X_2>2\theta_0),\ I(X_1+X_3>2\theta_0))$.

$\Rightarrow E_\theta(W_n^+) \simeq n^2\mu(\theta)$, $Var_\theta(W_n^+) \simeq n^3\sigma^2(\theta)$.

pf) (a) (1) $\dfrac{1}{\binom{n}{2}} W_n^+ = \dfrac{1}{\binom{n}{2}} \sum_{1\le i<j\le n} \underbrace{I(X_i+X_j > 2\theta_0)}_{h(X_1,X_2)} + \dfrac{1}{\binom{n}{2}} \sum_{i=1}^{n} I(X_i > \theta_0)$.

(* 위치모수모형)

$= U_n + O_p(\frac{1}{n})$.

(2) U_n의 점근정규성) $U_n = \tilde{U}_n + R_n$, $\tilde{U}_n = \mu(\theta) + \frac{2}{n}\sum_{i=1}^{n} h_1(X_i)$.

$h_1(x) = E_\theta[h(X_1,X_2) \mid X_1 = x] - \mu(\theta)$

$= P_\theta(X_1+X_2 > 2\theta_0 \mid X_1 = x) - \mu(\theta)$

$= P_\theta(x + Z_2 > 2\theta_0 - \theta \mid X_1 = x) - \mu(\theta)$

$= 1 - F(2\theta_0 - \theta - x) - \mu(\theta)$.

$\Rightarrow h_1(X_i) = 1 - F(2\theta_0 - \theta - X_i) - \mu(\theta)$, $X_i \sim f(x-\theta)$

$h_1(X_i') = 1 - F(2\theta_0 - 2\theta - X_i') - \mu(\theta)$. $X_i' \sim f(x)$.

then $\Rightarrow \sqrt{n}(U_n - \mu(\theta)) \xrightarrow{d} N(0,\ 4\,Var_\theta(h_1(X_i)))$

$h(X_1,X_2)$: bounded ftn

$\mu(\theta) = E_\theta[h(X_1,X_2)]$

$= E_\theta[h_1(X_1)] = E_\theta[1 - F(2\theta_0 - 2\theta - X_1')]$

$= \int (1 - F(2\theta_0 - 2\theta - x)) f(x)\, dx$

$\sigma^2(\theta) = 4\,Var_\theta(h_1(X_1))$

$\longrightarrow \sigma^2(\theta_0) = 4\, Var(\underbrace{1 - F(-X_1')}_{\overset{d}{=}\ U \sim U(0,1)}) = 4 \cdot \frac{1}{12} = \frac{1}{3}$

(3) W_n^+의 점근정규성) $\frac{1}{\binom{n}{2}} W_n^+ = U_n + O_p(\frac{1}{n})$

$$\Leftrightarrow \quad \sqrt{n}\left(\frac{1}{\binom{n}{2}} W_n^+ - \mu(\theta)\right)/\sigma(\theta) \simeq \frac{\sqrt{n}(U_n - \mu(\theta))}{\sigma(\theta)} \xrightarrow{d} N(0,1).$$

↳ 앞에서 $\sqrt{Var(X_1)}$ 값을 구했다.

Ex 9.3.7. (t-검정 ARE)

$$ARE(\{W_n^+\}, \{T_n\}) = \left[\frac{\dot{\mu}_1(\theta_0)}{\sigma_1(\theta_0)}\right]^2 \Big/ \left[\frac{\dot{\mu}_2(\theta_0)}{\sigma_2(\theta_0)}\right]^2$$

$$= 12\left\{\int f^2(x)dx\right\}^2 \cdot \int x^2 f(x)dx.$$

sol) $\dot{\mu}(\theta)\big|_{\theta=\theta_0} = \int 2 f(2\theta_0 - 2\theta - x) f(x)\, dx \,\big|_{\theta=\theta_0}$

$= 2\int f^2(x)dx$ (∵ 대칭성)

$\therefore slope_1 = \frac{4\{\int f^2(x)dx\}^2}{\frac{1}{3}} = \underbrace{12\left\{\int f^2(x)dx\right\}^2 \times \frac{1}{\sigma}}$ 에 제곱수 추가.

⇒ 오정한 분포가

	ARE
$N(\mu, \sigma^2)$	$\frac{3}{\pi}$
$L(\mu, \sigma)$	$\frac{\pi^2}{9}$
$DE(\mu, \sigma)$	1.5

Ex 추가 이차원 정규분포에서 Kendall's Tau.

$$Z_i = \begin{pmatrix} X_i \\ Y_i \end{pmatrix} \sim N_2\left[\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}\right]$$

통계량. $T_n = \frac{1}{\binom{n}{2}} \sum_{1\le i<j\le n} sgn((X_i - X_j)(Y_i - Y_j))$ → order-2의 U-통계량.

where $h(z_i, z_j) = sgn\{(X_i - X_j)(Y_i - Y_j)\}$.

sol). (1) $h_1(z) = h_1(x, y)$

$= E_\rho\left[h(Z_1, Z_2) \,\middle|\, Z_1 = \begin{pmatrix} x \\ y \end{pmatrix}\right]$

$= E_\rho[\, sgn(x - X_2)(y - Y_2)\,]$

$= P_\rho(X_2 > x, Y_2 > y) + P_\rho(X_2 < x, Y_2 < x) - P_\rho(X_2 > x, Y_2 < y) - P_\rho(X_2 < x, Y_2 > y).$

$= 4\Phi_2(x, y : \rho) - 2\Phi(x) - 2\Phi(y) + 1.$ ※ $\Phi_2(x, y; \rho) = P_\rho(X_2 \le x, Y_2 \le y)$

$h_1(Z_i) = 4\Phi_2(X_i, Y_i; \rho) - 2\Phi(X_i) - 2\Phi(Y_i) + 1.$

(2) $\mu(\rho) = E[h_1(Z_1)]$

$= 4E[\Phi_2(X, Y : \rho)] - 1$ $\quad \begin{matrix}\Phi(X_i) \\ \Phi(Y_i)\end{matrix} \sim Unif(0,1).$

$= \frac{2}{\pi}\arcsin(\rho)$

$$\sigma^2(0) = 4 \cdot \mathrm{Var}_0(h_1(Z))$$
$$= 4\ \mathrm{Var}\left(4\Phi(X_i)\Phi(Y_i) - 2\Phi(X_i) - 2\Phi(Y_i) + 1\right)$$
$$= 4\ \mathrm{Var}\left((2\Phi(X_i)-1)(2\Phi(Y_i)-1)\right) \qquad \begin{matrix} 2\Phi(X_i)-1 \\ 2\Phi(Y_i)-1 \end{matrix} \sim \mathrm{Unif}(-1,1).$$
$$= 4\ \mathrm{Var}(U_i \cdot V_i)$$
$$= 4\ E[U_i^2 V_i^2] \overset{*}{=} 4 \cdot \frac{1}{9}.$$

(3) 정리) $\sqrt{n}\left(T_n - \mu(\rho)\right) \xrightarrow{d} N\left(0,\ 4\,\mathrm{Var}_\rho(h_1(Z_i))\right)$

(4) ARE) $\dot{\mu}(\rho)\big|_{\rho=0} = \frac{2}{\pi} \cdot \frac{1}{\sqrt{1-\rho^2}}\Big|_{\rho=0} = \frac{2}{\pi}.$

$$\mathrm{slope}_\tau = \left(\frac{2}{\pi}\right)^2 \Big/ \frac{4}{9} = \frac{9}{\pi^2}. \quad \Rightarrow \mathrm{ARE} = \frac{9}{\pi^2} \ \text{우위}_{\hat\rho,\ \tau}$$

* 표본상관계수의 극한분포 (예 5.3.6.). $\begin{pmatrix} X_i \\ Y_i \end{pmatrix} \sim N\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}\right).$

$$\hat\rho_n = \frac{\sum (X_i - \bar X)(Y_i - \bar Y)}{\sqrt{\sum (X_i - \bar X)^2}\sqrt{\sum (Y_i - \bar Y)^2}} = \frac{\overline{XY} - \bar X \bar Y}{\sqrt{\overline{X^2} - (\bar X)^2}\sqrt{\overline{Y^2} - (\bar Y)^2}} = g(\bar X, \bar Y, \overline{XY}, \overline{X^2}, \overline{Y^2}).$$

(1) $Z_i = (X_i, Y_i, X_iY_i, X_i^2, Y_i^2)$ 이라 하면,

$$\sqrt{n}\left(\frac{1}{n}\sum Z_i - E(Z_1)\right) \xrightarrow{d} N_5\left(0, \mathrm{Var}(Z_1)\right). \quad \text{where } E(Z_1) = (0,0,\rho,1,1)^T \triangleq \theta.$$

δ-method

$$\sqrt{n}\left(\hat\rho_n - \rho\right) \xrightarrow{d} N\left(0,\ \underbrace{\dot g(\theta)^T \mathrm{Var}(Z_1)\, \dot g(\theta)}_{=V}\right)$$

where $g(\theta) = \rho$. $\dot g(\theta) = \left(0, 0, 1, -\frac{\rho}{2}, -\frac{\rho}{2}\right)^T.$

$$V = \mathrm{Var}\left(\dot g(\theta)^T Z_1\right) = \mathrm{Var}\left(X_1Y_1 - \frac{\rho}{2}X_1^2 - \frac{\rho}{2}Y_1^2\right) = (1-\rho^2)^2$$

(2) slope$_{\hat\rho}$ 계산) $\dot\mu(\rho)\big|_{\rho=0} = \frac{d}{d\rho}\rho = 1$, $\sigma(\rho)\big|_{\rho=0} = 1. \Rightarrow 1.$

$$* \quad g\Big(\underset{0}{m_{10}}, \underset{0}{m_{01}}, \underset{\rho}{m_{11}}, \underset{1}{m_{20}}, \underset{1}{m_{02}}\Big) = \frac{m_{11} - m_{10}m_{01}}{\sqrt{m_{20} - m_{10}^2}\sqrt{m_{02} - m_{01}^2}} = \frac{m_{11} - m_{10}m_{01}}{\sqrt{S_X}\sqrt{S_Y}}$$

$$\dot g(\quad \text{"} \quad) = \left(\frac{m_{10}}{S_X} g - \frac{m_{01}}{\sqrt{S_XS_Y}},\ \frac{m_{01}}{S_Y} g - \frac{m_{01}}{\sqrt{S_XS_Y}},\ \frac{1}{\sqrt{S_XS_Y}},\ -\frac{1}{2S_X} g,\ -\frac{1}{2S_Y} g\right)^T$$

$$* \quad \mathrm{Var}(\overbrace{X_1Y_1 - \tfrac{\rho}{2}X_1^2 - \tfrac{\rho}{2}Y_1^2}^{W}) = E[W^2] - (E(W))^2 = E[W^2] \quad \begin{pmatrix} E(W) \\ = \rho - \frac{\rho}{2}\cdot 2 = 0 \end{pmatrix}$$

$$= E\Big[\underbrace{X^2Y^2}_{1+2\rho^2} + \underbrace{\tfrac{\rho^2}{4}X^4}_{\frac{3}{4}\rho^2} + \underbrace{\tfrac{\rho^2}{4}Y^4}_{\frac{3}{4}\rho^2} - \underbrace{\rho X^3Y}_{-3\rho^2} - \underbrace{\rho XY^3}_{-3\rho^2} - \underbrace{\tfrac{\rho^2}{2}X^2Y^2}_{\frac{\rho^2}{2}+\rho^4}\Big]$$

$$= 1 - 2\rho^2 + \rho^4 = (1-\rho^2)^2.$$

$$\begin{cases} E(X^4) = E(Y^4) = 3 \\ E(X^3Y) = E(XY^3) = 3\rho \\ E(X^2Y^2) = 1 + 2\rho^2 \end{cases}$$