<수리통계 2>

6장. 추정.

6.1. 적률이용추정법.

6.1.1. 정의.

Def (Moments) $m_k = E(X^k)$.

(MME) $\hat{m}_k = \frac{1}{n}\sum_{i=1}^{n} X_i^k$

$$\eta = g(m_1 \dots m_k) \leftarrow \hat{\eta}^{MME} = g(\hat{m}_1, \dots, \hat{m}_k).$$

Ex (모분산의 MME)

$$\sigma^2 = E(X^2) - (E(X))^2 = m_2 - m_1^2$$

$$\hat{\sigma}^2 = \hat{m}_2 - \hat{m}_1^2 = \frac{1}{n}\Sigma X_i^2 - \left(\frac{1}{n}\Sigma X_i\right)^2 = \frac{1}{n}\Sigma(X_i - \bar{X})^2$$

Ex (적률이용 추정량의 비유일성)

$$\text{Poisson}(\mu) \Rightarrow E(X) = \mu,\quad Var(X) = \mu, = E(X^2) - (E(X))^2.$$

$$\Rightarrow \hat{\mu} = \hat{m}_1 = \hat{m}_2 - \hat{m}_1^2.$$

Ex (모상관계수의 MME)

$$\rho = Corr(X_1, X_2) = \frac{Cov(X_1, X_2)}{\sqrt{Var(X_1)}\sqrt{Var(X_2)}} = \frac{E(X_1X_2) - E(X_1)E(X_2)}{\sqrt{E(X_1^2) - (E(X_1))^2}\sqrt{E(X_2^2) - (E(X_2))^2}}$$

$$\Rightarrow \hat{\rho} = \frac{\hat{m}_{11} - \hat{m}_{10}\hat{m}_{01}}{\sqrt{\hat{m}_{20} - \hat{m}_{10}^2}\sqrt{\hat{m}_{02} - \hat{m}_{01}^2}} \quad \left(\hat{m}_{r,s} = \frac{1}{n}\Sigma X_{1i}^r X_{2i}^s\right)$$

6.1.2. 적률추정량의 일치성.

* Def (확률수렴)

$\forall \varepsilon > 0,\ P(|X_n - X| > \varepsilon) \to 0.$ as $n \to \infty$.

Thm. $\hat{\eta}^{MME} \xrightarrow{P} \eta.$

Ex $\hat{m}_r = \frac{1}{n}\Sigma X_i^r \triangleq \frac{1}{n}\Sigma Y_i. \xrightarrow{P} m_r.$, $E(X_1^{2r}) < \infty.$

pf) $P\left(\left|\frac{1}{n}\Sigma X_i^{2r} - E(X_1^{2r})\right| > \varepsilon\right) \le \frac{1}{n} Var(X_1^{2r}) \cdot \frac{1}{\varepsilon^2} \to 0.$

* (Chebyshev) $P(|X - EX| \ge k\sigma) \le \frac{1}{k^2}.$

6.1.3. 적률이용추정량의 극한 분포.

Thm $\sqrt{n}(\hat{\eta}_n - \eta) \xrightarrow{d} N(0, \sigma^2).$ $(\eta = g(\underbrace{m_1 \dots m_k}_{m}))$

where $\sigma^2 = (\dot{g}(m))^T \Sigma\, \dot{g}(m)$, $\Sigma = (m_{r+s} - m_r m_s).$

pf) $g(\hat{m}) - g(m) = \left(\frac{\partial}{\partial m_1} g(m) \cdots \frac{\partial}{\partial m_k} g(m)\right)^T \begin{pmatrix} \hat{m}_1 - m_1 \\ \vdots \\ \hat{m}_k - m_k \end{pmatrix} + \frac{1}{2}(\hat{m}-m)^T \left[\frac{\partial^2 g(m^*)}{\partial m_i \partial m_j}\right](\hat{m}-m)$

$\Leftrightarrow \sqrt{n}\,(g(\hat{m}) - g(m)) = \underbrace{\dot{g}(m)^T \sqrt{n}(\hat{m}-m)}_{①} + \underbrace{\frac{1}{2}\frac{1}{\sqrt{n}} \sqrt{n}(\hat{m}-m)^T \left[\frac{\partial^2 g(m^*)}{\partial m_i \partial m_j}\right](\hat{m}-m)}_{②}$

$① \xrightarrow{d} N(0,\ \dot{g}(m)^T \Sigma\, \dot{g}(m))$

$② = \underbrace{\frac{1}{2\sqrt{n}}}_{\xrightarrow{P} 0.} (\ \cdot\ ) \xrightarrow{d} N(0,\Sigma)^T \left[\frac{\partial^2 g(m^*)}{\partial m_i \partial m_j}\right] N(0,\Sigma). \xrightarrow{P} 0.$

Ex $\hat{\sigma}^{2\,MME} = \hat{m}_2 - \hat{m}_1^2$. $g(m_1, m_2) = m_2 - m_1^2$

$\sqrt{n}(\hat{\sigma}^2 - \sigma^2) \to N(0, \sigma^2)$.

$\sigma^2 = (-2m_1 \quad 1)\begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{bmatrix}\begin{pmatrix} -2m \\ 1 \end{pmatrix}$

$= 4m_1 \sigma_{11}^2 - 4m_1 \sigma_{12} + \sigma_{22}$

$= 4m_1(m_2 - m_1^2) - 4m_1(m_3 - m_1 m_2) + (m_4 - m_2^2)$.

6.1.4. Sample Properties of Order Statistics.

Thm 6.1.A. $X_{k_1:n} \ \cdots \ X_{k_r:n}$, $1 \le k_1 < k_2 \cdots < k_r \le n$

$\exists\ q_1 \ldots q_r$ s.t. $\sqrt{n}\left(\frac{k_i}{n} - q_i\right) \to 0$.

$\Rightarrow \sqrt{n}\left[(X_{k_1:n}, \ldots, X_{k_r:n})^T - (\xi_{q_1}, \ldots, \xi_{q_r})^T\right] \xrightarrow{d} N_r(0, \Sigma)$.

where ξ_{q_j} is the population q_j-th quantile $(F^{-1}(q_j) = \xi_{q_j})$

and $\Sigma = (\sigma_{ij})$, $\sigma_{ij} = \dfrac{q_i(1-q_j)}{f(\xi_{q_i})\, f(\xi_{q_j})}$

Thm 6.1.B. (Asymptotic Normality of IQR). * IQR : $X_{[\frac{3n}{4}]:n} - X_{[\frac{n}{4}]:n}$.

$\sqrt{n}(IQR - (\xi_{3/4} - \xi_{1/4})) \xrightarrow{d} N(0, V)$

where $V_{IQR} = \dfrac{3/16}{[f(\xi_{1/4})]^2} + \dfrac{3/16}{[f(\xi_{3/4})]^2} - \dfrac{2\cdot(1/16)}{f(\xi_{1/4})f(\xi_{3/4})}$.

pf) $\sqrt{n}\begin{bmatrix} X_{[\frac{n}{4}]:n} - \xi_{1/4} \\ X_{[\frac{3n}{4}]:n} - \xi_{3/4} \end{bmatrix} \xrightarrow{d} N\left(0, \begin{bmatrix} \frac{\frac{1}{4}\cdot\frac{3}{4}}{f(\xi_{1/4})^2} & \frac{\frac{1}{4}\cdot\frac{1}{4}}{f(\xi_{1/4})f(\xi_{3/4})} \\ \triangle & \frac{\frac{3}{4}\cdot\frac{1}{4}}{f(\xi_{3/4})^2} \end{bmatrix}\right) \stackrel{\triangle}{=} W$.

$\sqrt{n}(IQR - (\xi_{3/4} - \xi_{1/4})) \xrightarrow{d} N\left(0, (-1\ \ 1)\, W \begin{pmatrix} -1 \\ 1 \end{pmatrix}\right)$.

Thm 6.1.C. (Bahadur Representation).

$X_1 \dots X_n \overset{iid}{\sim}$ cdf F. $\in C^1$.

$\Rightarrow$ Sample quantile $F_n^{-1}(p) = \frac{1}{n}\sum Y_i + R_n$

where $Y_i = \dfrac{p - I(X_i \le \xi_p)}{f(\xi_p)}$, $R_n = o\left(\frac{1}{\sqrt{n}}\right)$.

6.1.5. Generalized Method of Moments.

Def $\mathbb{E}(g(X_i;\theta_0)) = 0$,

$\hat{g}_n(\theta) = \frac{1}{n}\sum g(X_i;\theta)$ 이고 True θ_0에 대해 $\hat{g}_n(\theta_0) \simeq 0$. (L.L.N.).

(추정량) $\hat{\theta}_n = \underset{\theta}{\mathrm{argmin}}\ \hat{g}_n(\theta)^T W_n \hat{g}_n(\theta)$. (혹은 $\frac{1}{n}\sum g(X_i;\theta) = 0$.)

Ex $X \perp \varepsilon \rightarrow E[X^T(y - X\beta)] = 0$.

$\Rightarrow \frac{1}{n}\sum X_i^T(y_i - X_i\beta) = 0$ 풀면 $\hat{\beta} = \left(\frac{1}{n}\sum X_i^T X_i\right)^{-1}\left(\frac{1}{n}\sum X_i^T y_i\right)$.

(점근적 성질). $\hat{\theta}_n = \mathrm{argmax}\left(\frac{1}{n}\sum h(X_i;\theta)\right)$

$\theta_0 = \mathrm{argmax}\left(E[h(X_i;\theta)]\right)$ $\Rightarrow \hat{\theta}_n \xrightarrow{P} \theta_0$.

(under some conditions).

6.2. 최대가능도 추정법.

6.2.1. 정의

Def $f(x;\theta)|\Delta x| = \prod_{i=1}^n f(x_i;\theta)|\Delta x_1|\cdots|\Delta x_n| \rightarrow$ 모수가 $\theta \in \Theta$ 일 때 관측값이 등장할 확률.

$\quad\ \ \| $

$L(\theta)$: 가능도 함수.

$\hat{\theta}^{MLE} = \underset{\theta\in\Theta}{\mathrm{argmax}}\ L(\theta)$.

$= \mathrm{argmax}\ \underbrace{l(\theta)}_{}$ $\log L(\theta)$.

6.2.2. 예제들

Ex $X_1 \dots X_n \overset{iid}{\sim} Poi(\lambda)$. $\lambda > 0$.

$L(\lambda) = \prod_{i=1}^n f(X_i;\lambda) = \prod_{i=1}^n \dfrac{\lambda^{x_i} e^{-\lambda}}{x_i!}$

$l(\lambda) = \sum \log f(X_i;\lambda) = \sum x_i \log\lambda - n\lambda +$ 상수.

$$\Rightarrow \hat{\lambda}^{MLE} = \begin{cases} 0 & \sum x_i = 0 \\ \bar{X} & \sum x_i > 0. \end{cases}$$

$\therefore \hat{\lambda}^{MLE} = \bar{X}$.

Ex $X_1 \dots X_n \overset{iid}{\sim}$ Bernoulli(p). $0 \le p \le 1$.

$L(p) = \prod p^{x_i} (1-p)^{1-x_i}$

(i) $0 < p < 1$.

$l(p) = \sum x_i \log p + (n - \sum x_i) \log(1-p)$.

$\dot{l}(p) = \frac{1}{p} \sum x_i - \frac{1}{1-p}(n - \sum x_i)$

$\ddot{l}(p) = -\frac{1}{p^2} \sum x_i - (n - \sum x_i) \frac{1}{(1-p)^2} < 0 \quad (\forall\, 0<p<1)$.

$\Rightarrow \hat{p}^{MLE} = \begin{cases} \bar{X} & 0 < \bar{X} < 1. \\ 0 & \bar{X} = 0 \to l(p) = n \log(1-p) \\ 1 & \bar{X} = 1 \to l(p) = n \log p. \end{cases}$

(ii) $p = 0$

$x_1 = \cdots = x_n = 0 \Rightarrow \bar{X} = 0, \quad \hat{p}^{MLE} = 0$.

(iii) $p = 1$

$x_1 = \cdots = x_n = 1 \Rightarrow \bar{X} = 1, \quad \hat{p}^{MLE} = 1$. $\quad \therefore \hat{p}^{MLE} = \bar{X}$

6.2.3. 정리 6.2.1, 정리 6.2.2.

Thm $\ddot{l}(\theta) < 0, \ \forall \theta \in \Omega_0$ 이고 $\dot{l}(\hat{\theta}) = 0, \ \hat{\theta} \in \Omega_0$ 이면 $\max_{\theta} l(\theta) = l(\hat{\theta})$.

pf) $l(\theta) = l(\hat{\theta}) + \cancel{\dot{l}(\hat{\theta})(\theta - \hat{\theta})} + \ddot{l}(\theta^*)(\theta - \hat{\theta})^2 < l(\hat{\theta})$. $\forall \theta \in \Omega_0$.

Thm $\ddot{l}(\theta) < 0. \ \forall \theta \in \Omega_0$ 이고 $\lim_{\theta \to \partial(\Omega_0)} l(\theta) = -\infty$ 이면 (끝점/ 무한히 커지거나 작아질).

$\exists$ unique $\hat{\theta}$ s.t. $\dot{l}(\hat{\theta}) = 0$.

pf) $\exists\, \hat{\theta}_1, \hat{\theta}_2$ s.t. $l(\hat{\theta}_1) = l(\hat{\theta}_2) = \max l(\theta)$.

$\Rightarrow l(\frac{1}{2}\hat{\theta}_1 + \frac{1}{2}\hat{\theta}_2) > \frac{1}{2} l(\hat{\theta}_1) + \frac{1}{2} l(\hat{\theta}_2), \quad \frac{1}{2}\hat{\theta}_1 + \frac{1}{2}\hat{\theta}_2 \in \Omega_0$. $\not\ni$.

6.2.4. MLE의 Invariance Property.

Thm. $\eta = g(\theta)$, g: θ의 일대일 함수 $\Rightarrow \hat{\eta}^{MLE} = g(\hat{\theta}^{MLE})$.

Ex $X_1 \dots X_n \overset{iid}{\sim} N(\mu, 1)$. $\eta = P(X_i > 0) = 1 - \Phi(-\mu)$의 MLE?

sol) $L(\mu) = \left(\frac{1}{\sqrt{2\pi}}\right)^n \exp\{-\sum \frac{1}{2}(X_i - \mu)^2\}$

$l(\mu) =$ 상수 $- \frac{1}{2} \sum (X_i - \mu)^2 =$ 상수 $+ n\bar{X}\mu - \frac{1}{2}\mu^2$

$\hat{\mu}^{MLE} = \bar{X}$. $\quad \hat{\eta}^{MLE} = 1 - \Phi(-\bar{X})$. $\quad X_i^2 -$

Ex $X_1 \dots X_n \overset{iid}{\sim} Exp(\theta)$. $\theta = \frac{1}{\lambda}$ 의 MLE?

sol) $f(x;\theta) = \frac{1}{\theta}\exp\left(-\frac{1}{\theta}x\right)$ → $\ell(\theta)$가 순오목이 아님.

$f(x;\lambda) = \lambda \exp(-\lambda x)$.

$\ell(\lambda) = n\log\lambda - \lambda\sum x_i$.

$\Rightarrow \hat{\lambda}^{MLE} = 1/\bar{X}$, $\hat{\theta}^{MLE} = 1/\hat{\lambda}^{MLE} = \bar{X}$.

6.2.5. 지수족에서의 최대가능도 추정량.

Def $f(x;\eta) = \exp\{\eta T(x) - A(\eta) + C(x)\}$, $x \in \underset{\text{support}}{\mathcal{X}}$, $\eta \in \underset{\text{모수공간}}{N}$.

Thm 조건 (i) $\mathcal{X}$가 모수에 의존하지 X.

(ii) N이 열린 집합을 포함. (iii) $T(x)$가 상수가 아님.

$X_1 \dots X_n \overset{iid}{\sim} f(x;\eta)$ 이면

$\dot{A}(\eta) = \frac{1}{n}\sum T(x_i)$ 혹은 $E[T(\eta)] = \frac{1}{n}\sum T(x_i)$ 의 근 $\hat{\eta}$는 $\hat{\eta}^{MLE}$.

pf) $T(X_1)$ 의 $mgf(s) = E\left(e^{sT(X_1)}\right) = \int e^{sT(x)} f(x;\eta)\,dx$

$= \int f(x; s+\eta)\,dx \cdot \exp(A(s+\eta) - A(\eta))$

$cgf(s) = A(s+\eta) - A(\eta)$.

$E[T(X_1)] = \left[\frac{d}{ds} cgf(s)\right] = \dot{A}(\eta)$

$Var[T(X_1)] = \left[\frac{d^2}{ds^2} cgf(s)\right] = \ddot{A}(\eta) \rightarrow \ddot{A}(\eta) > 0$.

$L(\eta) = \exp\{\eta\sum T(X_i) - nA(\eta) + \sum C(X_i)\}$. $\Rightarrow \ddot{\ell}(\eta) = -n\ddot{A}(\eta) < 0$.

$\frac{d\ell(\eta)}{d\eta} = \sum T(X_i) - n\dot{A}(\eta) = 0 \iff \dot{A}(\eta) = \frac{1}{n}\sum T(X_i)$.

↳ $\hat{\eta}^{MLE}$.

Ex ① Bernoulli (p). $0<p<1$.

* $Bin(n,p) \rightarrow \binom{n}{x}p^x(1-p)^{1-x}$ 에서 $\binom{n}{x}$는 상수이므로 $L(p)$는 동일하다.

$f(x;p) = p^x(1-p)^{1-x} = \exp\{x\log p + (1-x)\log(1-p)\}$

$= \exp\{x(\log\frac{p}{1-p}) + \log(1-p)\}$.

Let $\eta = \log\frac{p}{1-p}$.

$f(x;\eta) = \exp\{\eta x - \log(1+e^\eta)\} \Rightarrow \begin{cases} T(x) = x, \\ A(\eta) = \log(1+e^\eta). \end{cases}$

$\frac{1}{n}\sum x_i = \dot{A}(\hat{\eta})$ 이고,

$\dot{A}(\eta) = \frac{e^\eta}{1+e^\eta} = p$. $\Rightarrow \hat{p} = \frac{1}{n}\sum x_i = \bar{X}$.

(2) $Poi(\lambda)$. $\lambda > 0$.

$$f(x;\lambda) = \frac{e^{-\lambda}\lambda^{x}}{x!} = \exp\{-\lambda + x\log\lambda - \log x!\}.$$

Let $\eta = \log\lambda$,

$$f(x;\eta) = \exp\{\eta x - e^{\eta} - \log x!\}. \Rightarrow \begin{cases} T(x) = x \\ A(\eta) = e^{\eta}. \quad \dot{A}(\eta) = e^{\eta}. \end{cases}$$

가능도방정식: $\frac{1}{n}\sum x_i = E(X) = \lambda \quad \Rightarrow \quad \hat{\lambda} = \overline{X}.$

or $\frac{1}{n}\sum x_i = e^{\eta} = \lambda.$

(3) $Geo(p)$. $0<p<1$.

$$f(x:p) = (1-p)^{x-1}p. = \exp\{(x-1)\log(1-p) + \log p\}$$
$$= \exp\{x\log(1-p) - \log(1-p) + \log p\}.$$

Let $\eta = \log(1-p)$

$$f(x;\eta) = \exp\{\eta x - \eta + \log(1-e^{\eta})\}. \Rightarrow \begin{cases} T(x) = x. \\ A(\eta) = \eta. \end{cases}$$

가능도방정식: $\frac{1}{n}\sum x_i = E(X) = 1/p \quad \Rightarrow \quad \hat{p} = 1/\overline{X}$

(4) $Exp(\theta)$. $\theta > 0$.

$$f(x:\theta) = \frac{1}{\theta}\exp(-\frac{1}{\theta}x) = \exp\{\underbrace{-\log\theta}_{\log\frac{1}{\theta} = \log(-\eta)} - \frac{1}{\theta}x\}.$$

Let $\eta = -\frac{1}{\theta}$.

$$f(x;\eta) = \exp\{\eta x + \log(-\eta)\}. \Rightarrow \begin{cases} T(x) = x. \\ A(\eta) = -\log(-\eta). \end{cases}$$

가능도방정식: $\frac{1}{n}\sum x_i = \theta. \quad \Rightarrow \quad \hat{\theta} = \overline{X}$

(5) $Pareto(1, \theta)$, $\theta > 0$.

$$f(x:\theta) = \theta x^{-\theta-1} = \exp\{(-\theta-1)\log x + \log\theta\}.$$
$$= \exp\{-\theta\log x + \log\theta - \log x\}.$$

$\eta = \theta$ 이므로

$$\begin{cases} T(x) = -\log x. \\ A(\eta) = -\log\eta. \rightarrow \dot{A}(\eta) = -\frac{1}{\eta}. \end{cases}$$

가능도방정식: $\frac{1}{n}\sum(-\log x_i) = -\frac{1}{\theta}. \quad \rightarrow \quad \hat{\theta} = 1/\overline{\log X}.$

(6) $X_1 \dots X_n \overset{iid}{\sim} N(\mu, 1)$.

(i) 일반적인 경우: $\hat{\mu}^{MLE} = \overline{X}$ (6.2.4).

(ii) $\mu \geq 0$

$$\hat{\mu}^{MLE} = \max(\overline{X}, 0)$$

(극한분포) $\mu > 0$: $\sqrt{n}(\hat{\mu}-\mu) \xrightarrow{d} N(0,1)$.

$\mu = 0$: $P_{\mu=0}(\sqrt{n}\hat{\mu} \le x) = \begin{cases} 0 & x<0. \\ 1/2 & x=0 \\ \Phi(x) & x>0. \end{cases}$

(iii) $\mu \in \mathbb{N}$. (discrete param. space).

$\ell(\mu) = n\bar{X}\mu - \frac{n}{2}\mu^2 +$ 상수.

$\rightarrow$ $\hat{\mu}^{MLE} = \bar{X}$의 가장 가까운 정수.

(점근적 성질) $P(\hat{\mu} \ne \mu) = P(|\bar{X}-\mu| > \frac{1}{2})$

$\rightarrow 0$ $\quad (\because \sqrt{n}(\bar{X}-\mu) \xrightarrow{d} N(0,1))$

$\Rightarrow$ $P(|\hat{\mu}-\mu| > \varepsilon) \le P(\hat{\mu} \ne \mu) \rightarrow 0$ 이므로 $\hat{\mu} \xrightarrow{P} \mu$.

⑦ (i) $X_1 \dots X_n \overset{iid}{\sim}$ Uniform $[0, \theta]$. $0<\theta<\infty$.

$L(\theta) = f(X_1 \dots X_n ; \theta) = \prod_{i=1}^{n} \frac{1}{\theta} I(0 \le X_i \le \theta)$

$= \frac{1}{\theta} I(0 < X_{(1)}) I(X_{(n)} \le \theta)$.

$\Rightarrow$ $\hat{\theta}^{MLE} = X_{(n)}$.

(점근분포) $n(\theta - X_{(n)}) \xrightarrow{d} Exp(1)$.

pf) $P(n(\theta - X_{(n)}) \le t) = P(X_{(n)} \ge \theta - \frac{t}{n})$

$= 1 - P(X_{(n)} \le \theta - \frac{t}{n})$

$= 1 - \left(\frac{\theta - \frac{t}{n}}{\theta}\right)^n \rightarrow 1 - e^{-t}$.

(ii) $X_1 \dots X_n \overset{iid}{\sim}$ Uniform$[\theta-1, \theta+1]$, $-\infty<\theta<\infty$. $\rightarrow$ non-unique MLE.

$L(\theta) = f(X_1 \dots X_n ; \theta) = \prod \frac{1}{2} I(\theta-1 \le x_i \le \theta+1)$

$= 2^{-n} I(\theta-1 \le x_{(1)}) I(x_{(n)} \le \theta+1)$

$\Rightarrow$ $[x_{(1)}+1, x_{(n)}-1]$ 의 모든 점에서 최댓값 2^{-n}.

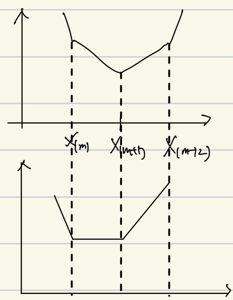
$\hookrightarrow \hat{\theta}^{MLE}$.

⑧ $X_1 \dots X_n \overset{iid}{\sim} DE(\theta, 1)$. : 이중지수분포

$f(x ; \theta) = e^{-|x-\theta|}$

$\ell(\theta) =$ 상수 $- \sum_{i=1}^{n} |x_i - \theta|$

$\begin{cases} n = 2m+1 \rightarrow \hat{\theta}^{MLE} = X_{(m+1)}. \\ n = 2m \rightarrow \hat{\theta}^{MLE} \in [X_{(m)}, X_{(m+1)}]. \end{cases}$



⑨ $X_1 \dots X_n \overset{iid}{\sim}$ Cauchy $(\theta, 1)$ $(\sim t(df=1))$.

$$f(x:\theta) = \frac{1}{\pi}\frac{1}{1+(x-\theta)^2}$$

$$L(\theta) = \prod_{i=1}^{n} \frac{1}{1+(x_i-\theta)^2}\cdot\frac{1}{\pi^n}.$$

$$l(\theta) = \Sigma -\log(1+(x_i-\theta)^2) + \text{상수}.$$

$$\dot{l}(\theta) = \Sigma \frac{2(x_i-\theta)}{1+(x_i-\theta)^2} = \Sigma\left[2(x_i-\theta)\prod_{j\neq i}\{1+(x_j-\theta)^2\}\right]\times\frac{1}{\prod(1+(x_k-\theta)^2)}$$

↳ θ에 대한 $1+2(n-1)$차 다항식 ⇒ N-R

6.3. 다차원 모수의 최대가능도 추정량

6.3.1. 정의

Def $X_1 \dots X_n \overset{iid}{\sim} f(x:\underline{\theta})$, $\underline{\theta} = (\theta_1, \dots, \theta_k)^T$

$$L(\underline{\theta}: x_1 \dots x_n) = f(x_1 \dots x_n : \underline{\theta}).$$

6.3.2. 예제들

Ex ① $X_1 \dots X_n \overset{iid}{\sim} DE(\mu, \sigma)$. $\sigma > 0$, $\mu \in \mathbb{R}$.

$$f(x;\mu,\sigma) = \frac{1}{2\sigma}\exp\left\{-\left|\frac{x-\mu}{\sigma}\right|\right\}.$$

$$L(\mu,\sigma) = \frac{1}{(2\sigma)^n}\exp\left\{-\frac{1}{\sigma}\Sigma|x_i-\mu|\right\}$$

$$l(\mu,\sigma) = -\frac{1}{\sigma}\Sigma|x_i-\mu| - n\log\sigma + \text{상수}.$$

⇒ for any σ, $\underset{\mu}{\operatorname{argmax}}\, l(\mu,\sigma) = \text{median}(X_i)$.

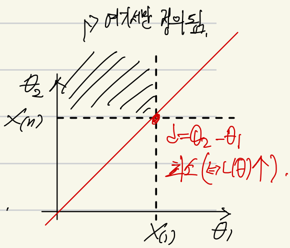
$$\therefore \hat{\mu}^{MLE} = \begin{cases} X_{(m+1)} & n=\text{odd} \\ \in [X_{(m)}, X_{(m+1)}] & n=\text{even}. \end{cases}$$

$$\& \ \hat{\sigma}^{MLE} = \underset{\sigma}{\operatorname{argmax}}\, l(\hat{\mu}^{MLE}, \sigma) = \frac{1}{n}\Sigma|X_i - \hat{\mu}|.$$

② $X_1 \dots X_n \overset{iid}{\sim}$ Uniform $[\theta_1, \theta_2]$. $\theta_1 < \theta_2$.

$$L(\theta_1, \theta_2) = \frac{1}{(\theta_2-\theta_1)^n}\prod_{i=1}^{n} 1(\theta_1 \le X_i \le \theta_2).$$

$$= \frac{1}{(\theta_2-\theta_1)^n}\, \Pi\, I(\theta_1 \le X_{(1)})\, I(X_{(n)} \le \theta_2).$$



$$\Rightarrow \hat{\theta}_2^{MLE} = X_{(n)}, \quad \hat{\theta}_1^{MLE} = X_{(1)}.$$

③ $X_1 \dots X_n \overset{iid}{\sim} N(\mu, \sigma^2)$.

$$l(\mu, \sigma^2) = -\frac{n}{2}\log\sigma^2 - \frac{1}{2\sigma^2}\Sigma(x_i-\mu)^2 + \text{상수}.$$

$\Sigma(x_i-\mu)^2 = \Sigma(x_i-\bar{X})^2 + n(\bar{X}-\mu)^2$ 이므로

$\underset{\mu}{\text{argmax}}\ l(\mu,\sigma^2) = \bar{X}. \Rightarrow \hat{\mu}^{MLE}$

$\underset{\sigma^2}{\text{argmax}}\ l(\hat{\mu},\sigma^2) = \frac{1}{n}\Sigma(x_i-\mu)^2 \Rightarrow \hat{\sigma}^{2\,MLE}.$

6.3.3. 모수의 함수의 최대가능도 추정.

Thm $\eta=(\eta_1,\eta_2)^T = (g_1(\theta), g_2(\theta))^T$ 가 H $\Rightarrow \hat{\eta}_1^{MLE} = g_1(\hat{\theta}^{MLE})$.

↳ H이 되도록 g_2 찾는다.

Ex ① $X_1 \ldots X_n \overset{iid}{\sim} N(\mu,\sigma^2)$.

Let $\eta = (|\mu|, sgn(\mu), \sigma^2)^T \rightarrow \theta=(\mu,\sigma^2)^T$의 H 변환.

$\rightarrow \hat{\eta}^{MLE} = (|\bar{X}|, sgn(\bar{X}), \frac{1}{n}\Sigma(X_i-\bar{X})^2)^T.$

↳ $\hat{\eta}_1^{MLE}$.

② $X_1 \ldots X_n \overset{iid}{\sim} \text{Bernoulli}(p)$.

Let $\eta = (p(1-p), sgn(p-\frac{1}{2}))^T \rightarrow p$의 H 변환.

$\Rightarrow \hat{\eta}_1^{MLE} = \hat{p}(1-\hat{p})$ where $\hat{p} = \frac{1}{n}\Sigma X_i$.

6.3.4. 정리 6.3.2. 정리 6.3.3.

Thm $c^T \ddot{l}(\theta) c < 0 \ \forall c\neq 0, \forall\theta\in\Omega_0$ 이고 $\dot{l}(\hat{\theta})=0$ 이면 $\max l(\theta) = l(\hat{\theta})$.

Thm $c^T \ddot{l}(\theta) c < 0 \ \forall c\neq 0, \forall\theta\in\Omega_0$ 이고 $\lim_{\theta\to\partial(\Omega_0)} l(\theta) = -\infty$ 이면 $\hat{\theta}$ uniquely exists.

6.3.5. 다중모수 지수족에서의 최대가능도 추정량

Thm $f(x;\eta) = \exp\{\sum_{j=1}^{k}\eta_j T_j(x) - A(\eta) + S(x)\}$ $x\in\mathcal{X}, \eta=(\eta_1\ldots\eta_k)^T\in N$.

조건 (i) $\mathcal{X}$가 η에 의존 X (ii) N이 k차원 open ball 포함 (iii) $T_j(x)\neq$ 상수.

가능도방정식 $\dot{A}(\eta) = \frac{1}{n}\Sigma T(x_i)$ / $E_\eta(T_j(X_i)) = \frac{1}{n}\Sigma T_j(x_i)$

↳ 의 해 $\hat{\eta} = \hat{\eta}(x_1\ldots x_n)$은 η^{MLE}.

↳ $= g(\hat{\theta}) = (g_1(\hat{\theta}) \ldots g_k(\hat{\theta}))^T$.

Ex ① $X_1 \ldots X_n \sim N(\mu,\sigma^2)$. $\theta = (\mu,\sigma^2)^T$.

$f(x;\theta) = \frac{1}{\sqrt{2\pi\sigma^2}}\exp\{-\frac{1}{2}\frac{(x-\mu)^2}{\sigma^2}\}. = \exp\{-\frac{1}{2\sigma^2}(x-\mu)^2 - \frac{1}{2}\log\sigma^2 - \frac{1}{2}\log 2\pi\}.$

$(\eta_1 = \frac{\mu}{\sigma^2}, \eta_2 = -\frac{1}{2\sigma^2})$ 로 일대일 변환.

$$-\frac{1}{2\sigma^2}(x^2-2\mu x+\mu^2) = \frac{\mu}{\sigma^2}x - \frac{1}{2\sigma^2}x^2 - \frac{\mu^2}{2\sigma^2}.$$

$$f(x;\eta) = \exp\{\eta_1 x + \eta_2 x^2 - \underbrace{A(\eta)}_{-\frac{\eta_1^2}{4\eta_2} - \frac{1}{2}\log(-\frac{\eta_2}{\pi})}\}.$$

$\Rightarrow E(X_1) = \frac{1}{n}\Sigma x_i = \mu,\quad E(X_1^2) = \frac{1}{n}\Sigma x_i^2 = \mu^2+\sigma^2.$ 이므로

$\hat{\mu}^{MLE} = \bar{X},\quad \hat{\sigma}^{2\,MLE} = \frac{1}{n}\Sigma(X_i-\bar{X})^2.$

② (이변량 정규분포) $\begin{pmatrix} X \\ Y \end{pmatrix} \sim N\left(\begin{pmatrix}\mu_1\\ \mu_2\end{pmatrix}, \begin{pmatrix}\sigma_1^2 & \sigma_1\sigma_2 \\ \sigma_1\sigma_2 & \sigma_2^2\end{pmatrix}\right).$

$f(x,y;\theta) = \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}}\exp(-\frac{1}{2}Q).\quad \cdots$

$$\begin{cases} T_1(x,y)=x, & T_2(x,y)=y, \quad T_3(x,y)=x^2, \\ T_4(x,y)=xy, & T_5(x,y)=y^2, \quad \eta=(\mu_1,\mu_2,\sigma_1^2,\sigma_2^2,\rho)^T. \end{cases}$$

$\Rightarrow E_\theta(X) = \frac{1}{n}\Sigma X_i,\quad E(Y) = \frac{1}{n}\Sigma Y_i,\ \cdots$

③ (다항분포). $X=(X_1\cdots X_k)^T \sim Multi(n, (p_1\cdots p_k)^T).$

제약조건 $\begin{cases} X_1+\cdots+X_k = n. \\ p_1+\cdots+p_k = 1. \quad 0\le p_j\le 1. \end{cases}$

$$f(x;p) = \binom{n}{x_1\cdots x_k} p_1^{x_1}\cdots p_k^{x_k}.$$

$$= \frac{n!}{x_1!\cdots x_k!}\prod_{j=1}^{k-1} p_j^{x_j}\cdot \underbrace{(1-p_1\cdots-p_{k-1})^{n-\sum_{j=1}^{k-1}x_j}}_{(n-\Sigma x_j)\log p_k}$$

$$= \exp\left\{\sum_{j=1}^{k-1} x_j\log p_j/p_k + n\log p_k + \log\frac{n!}{\Pi x_i!}\right\}.$$

$$= \exp\{\Sigma x_j\eta_j - nA(\eta) + c(x)\}.$$

$$\Rightarrow \begin{cases} \eta_j = \log\frac{p_j}{p_k} = \log\frac{p_j}{1-p_1\cdots-p_{k-1}} & j=1\ldots k-1. \\ A(\eta) = -\log(1-p_1-\cdots-p_k). \\ T_j(X) = x_j. \end{cases}$$

$(X_1\ldots X_k)^T$는 $\sum_{i=1}^{n} Y_i$, $Y_i=(Y_{i1}\ldots Y_{ik})^T \sim Multi(1,(p_1\cdots p_k)^T).$

가능도방정식: $E_\eta(T_j(X)) = \frac{1}{n}\Sigma T_j(X_i). = \frac{1}{n}\Sigma Y_{ij}.$

$\Leftrightarrow\ \hat{p}_j^{MLE} = x_j/n.\ \ \&\ \ \hat{p}_k^{MLE} = 1-\hat{p}_1-\cdots-\hat{p}_{k-1} = \frac{x_k}{n}.$

6.4. 최대가능도 추정량의 점근적 성질.

6.4.1. 예제들: 일치성, 극한분포.

Ex ① Bernoulli (p): $\hat{p}^{MLE} = \bar{X}.$

(i) $\hat{p}^{MLE} = \hat{p}^{MME} \xrightarrow{P} p$.

(ii) By CLT, $\sqrt{n}(\hat{p}^{MLE} - p) \xrightarrow{d} N(0, p(1-p))$.

② Poisson (μ) : $\hat{\mu}^{MLE} = \bar{X}$.

(i) $\hat{\mu}^{MLE} = \hat{\mu}^{MME} \xrightarrow{P} \mu$.

(ii) By CLT, $\sqrt{n}(\hat{\mu}^{MLE} - \mu) \xrightarrow{d} N(0, \mu)$.

③ Exp (θ) : $\hat{\theta}^{MLE} = \bar{X}$.

(i) $\hat{\theta}^{MLE} = \hat{\theta}^{MME} \xrightarrow{P} \theta$.

(ii) By CLT, $\sqrt{n}(\hat{\theta}^{MLE} - \theta) \xrightarrow{d} N(0, \theta^2)$.

④ $N(\mu, \sigma^2)$: $\hat{\mu}^{MLE} = \bar{X}$, $\hat{\sigma^2}^{MLE} = \frac{1}{n}\sum(X_i - \bar{X})^2$

(i) $\hat{\mu}^{MLE} = \hat{\mu}^{MME} \xrightarrow{P} \mu$, $\hat{\sigma^2}^{MLE} = \hat{\sigma^2}^{MME} \xrightarrow{P} \mu$.

(ii) By thm, $\sqrt{n}(\hat{\mu}^{MLE} - \mu) \sim N(0, \sigma^2)$.

By thm, $n\hat{\sigma}^2/\sigma^2 \sim \chi^2(n-1)$.

thm, $\sqrt{n}(\hat{\sigma}^2 - \sigma^2) \xrightarrow{d} N(0, 2\sigma^4)$

⑤ $N(\mu_1, \mu_2, \sigma_1^2, \sigma_2^2, \rho)$.

(i) $\hat{\rho}^{MLE} = \hat{\rho}^{MME} \xrightarrow{P} \rho$.

(ii) By δ-method, $\sqrt{n}(\hat{\rho} - \rho) \xrightarrow{d} N(0, (1-\rho^2)^2)$.

⑥ $X_1 \dots X_n \overset{iid}{\sim} Exp(\mu, \sigma)$, $f(x) = \frac{1}{\sigma}\exp\left(-\frac{(x-\mu)}{\sigma}\right)$. $x \geq \mu$.

$\rightarrow$ $\hat{\mu}^{MLE} = X_{(1)}$, $\hat{\sigma}^{MLE} = \bar{X} - X_{(1)}$.

$$\begin{cases} n(\hat{\mu} - \mu) \xrightarrow{d} Exp(\sigma) \\ \sqrt{n}(\hat{\sigma} - \sigma) \xrightarrow{d} N(0, \sigma^2). \end{cases}$$

6.4.2. 최대가능도 추정량의 일치성

$X_1 \dots X_n \overset{iid}{\sim} f(x_i; \theta)$. $\rightarrow$ $\hat{\theta}_n^{MLE} = \underset{\theta}{\operatorname{argmax}}\, l_n(\theta)$

$= \operatorname{argmax}\, \bar{l}_n(\theta)$.

$\bar{l}_n(\theta) = \frac{1}{n}\sum \log f(X_i; \theta) \xrightarrow{P} E_{\theta_0}[\log f(X; \theta)]$

↓ def ↓ K-L Divergence

$\hat{\theta}_n = \operatorname{argmax}\, \bar{l}_n(\theta) \xrightarrow{P?} \theta_0 = \operatorname{argmax} E_{\theta_0}[\log f(X; \theta)]$

6.4.3. Kullback-Leibler Divergence.

Thm Let $KL(\theta, \theta^0) = -E_{\theta_0}\left[\log \frac{f(X:\theta)}{f(X:\theta_0)}\right]$

조건 $\boxed{R0}$ (식별가능성) $f(x:\theta) = f(x:\theta^0) \Rightarrow \theta = \theta^0$.

$\boxed{R1}$ (공통의 토대) $\mathcal{X} = \{x : f(x:\theta) > 0\}$이 θ에 의존 X

$\Rightarrow$ (i) $KL(\theta, \theta^0) \geq 0 \quad \forall \theta \in \Omega$

(ii) $KL(\theta, \theta^0) = 0 \iff \theta = \theta^0$. <Jensen Ineq.>

pf) $\left[E[\phi(X)] \geq \phi(E[X]) \quad \phi: \text{convex}. \quad \text{'='} \text{ when } \phi(X) = c.\right]$ $\to \phi(t) = \log t$

$KL(\theta, \theta^0) = E_{\theta_0}\left[-\log \frac{f(X:\theta)}{f(X:\theta_0)}\right]$

$\geq -\log\left(E_{\theta}\left[\frac{f(X:\theta)}{f(X:\theta_0)}\right]\right) = 0.$

$\to \int \frac{f(x:\theta)}{f(x:\theta_0)} \cdot f(x:\theta_0)\, dx = 1$

등호 : $\frac{f(X:\theta)}{f(X:\theta_0)}$ 이 상수 $\iff \theta = \theta_0$ (by R0).

- By Thm, $\theta_0 = \text{argmax}\, E_{\theta_0}[\log f(x:\theta)]$.

6.4.4. 최대가능도 추정량의 일치성.

조건 (i) (균등 확률수렴성). $\sup_{|\theta-\theta_0| \leq k} |\bar{l}_n(\theta) - E_{\theta_0}[\log f(x:\theta)]| \xrightarrow{P} 0$.

(ii) (기대로그가능도의 연속성) $E[\log f(x:\theta)]$가 θ에 대한 연속.

(i)' (로그 우도 함수 오목성) $l_n(\theta)$ 이 strictly concave. $\Rightarrow$ (i) 성립.

Ex Gamma(α, β)

$f(x:\theta) = \frac{1}{\Gamma(\alpha)\beta^\alpha} x^{\alpha-1} \exp(-\frac{x}{\beta}), \quad x > 0.$

$= \exp\{(\alpha-1)\log x - \frac{1}{\beta}x - \log\Gamma(\alpha) - \alpha\log\beta\}$. $\to$ 지수족.

$\begin{cases} T_1(x) = \log x, & T_2(x) = x. \\ \eta_1(\alpha,\beta) = \alpha - 1, & \eta_2(\alpha,\beta) = -\frac{1}{\beta}. \end{cases}$ $\Rightarrow$ 조건 (i)', (ii) 만족.

$\partial l(\theta)/\alpha = 0, \partial l(\theta)/\beta = 0 \Rightarrow \hat{\alpha}, \hat{\beta}$는 일치성.

6.4.5. 최대가능도 추정량의 점근 정규성.

(1) 스케치

$\bar{l}_n(\theta) = \frac{1}{n} l_n(\theta) = \frac{1}{n}\sum \log f(x:\theta). \to \dot{\bar{l}}_n(\hat{\theta}) = 0$

$0 = \dot{\bar{l}}_n(\hat{\theta}_n) \simeq \dot{\bar{l}}_n(\theta) + \ddot{\bar{l}}_n(\theta)(\hat{\theta} - \theta)$ $\qquad \theta$ = true.

$\Leftrightarrow \hat{\theta} - \theta = [-\ddot{\bar{l}}_n(\theta)]^{-1} \dot{\bar{l}}_n(\theta)$.

$$\Leftrightarrow \quad \sqrt{n}(\hat{\theta}-\theta) \simeq \underbrace{[-\ddot{l}_n(\theta)]^{-1}}_{①} \underbrace{\sqrt{n}\,\dot{l}_n(\theta)}_{②}.$$

$$① = -\frac{1}{n}\sum \frac{\partial^2}{\partial\theta^2}\log f(x_i:\theta) = \frac{1}{n}\sum\Big[\underbrace{-\frac{\partial^2}{\partial\theta^2}\log f(x_i:\theta)}_{U_i}\Big]$$

$$\xrightarrow{P} E[U_i] = E\Big[-\frac{\partial^2}{\partial\theta^2}\log f(x_i:\theta)\Big] = I(\theta).$$

$$② = \sqrt{n}\,\frac{1}{n}\sum\frac{\partial}{\partial\theta}\log f(x_i:\theta) = \frac{1}{\sqrt{n}}\sum\underbrace{\frac{\partial}{\partial\theta}\log f(x_i:\theta)}_{W_i}.$$

$$W_i \sim \Big(E\Big[\frac{\partial}{\partial\theta}\log f(x:\theta)\Big],\ Var\Big[\frac{\partial}{\partial\theta}\log f(x:\theta)\Big]\Big)$$

$$\Rightarrow \text{By CLT,}\quad \frac{1}{\sqrt{n}}\sum W_i \xrightarrow{d} N\Big(\underbrace{E\Big[\frac{\partial}{\partial\theta}\log f(x_i:\theta)\Big]}_{=0},\ Var\Big[\frac{\partial}{\partial\theta}\log f(x_i:\theta)\Big]\Big).$$

$$\Rightarrow \quad \sqrt{n}(\hat{\theta}-\theta) \simeq [I(\theta)]^{-1}\sqrt{n}\,\dot{l}_n(\theta) \xrightarrow{d} N\Big(0,\ [I(\theta)]^{-2}\,Var\Big[\frac{\partial}{\partial\theta}\log f(x:\theta)\Big]\Big).$$

(2) 조건들

$\boxed{R0}$ $\boxed{R1}$: 일치성

$\boxed{R2}$ (열린 모수 공간) Ω : k차원 open set.

$\boxed{R3}$ (미분 가능한 로그가능도). 1·2차 편도함수 존재 & 연속.

$\boxed{R4}$ (적분·미분 순서 교환 가능) $E_\theta(U(X))$ 존재. $\partial^r E_\theta(U(X)) = E_\theta(\partial U(X))$.

$\boxed{R5}$ (정보량의 존재) $E\Big[-\frac{\partial^2}{\partial\theta^2}\log f(x_i:\theta)\Big]$ 존재.

$\boxed{R6}$ (최대가능도 추정량의 일치성)

$\boxed{R7}$ (잉여항의 크기). $\max_{\theta\in\Theta}\Big\|\frac{\partial^3}{\partial\theta^3}\log f(x:\theta)\Big\| \le M(x)$. $E_\theta(M(X)) < \infty$. $\forall\theta$.

(3) Score Function 의 성질 : $\dot{l}(\theta)$

Thm $\boxed{R0} \sim \boxed{R5}$

$\Rightarrow$ (i) $E_\theta\Big[\frac{\partial}{\partial\theta}\log f(X:\theta)\Big] = 0.$

(ii) $I(\theta) \triangleq Var\Big(\frac{\partial}{\partial\theta}\log f(X:\theta)\Big) = E\Big[-\frac{\partial^2}{\partial\theta^2}\log f(X:\theta)\Big]$

pf) (i) $\int \frac{\partial}{\partial\theta}\log f(x:\theta)\cdot f(x:\theta)\,dx$

$= \int \frac{\partial}{\partial\theta}f(x:\theta)\,/\,f(x:\theta) \times f(x:\theta)\,dx$

$= \int \frac{\partial}{\partial\theta}f(x:\theta)\,dx = \frac{\partial}{\partial\theta}\int f(x:\theta)\,dx = 0.$

(ii) $\int f(x:\theta)\,dx = 1 \Leftrightarrow \int \exp(\log f(x:\theta))\,dx$

미분 $\downarrow$ $\int \frac{\partial}{\partial\theta}\log f(x:\theta)\exp(\log f(x:\theta))\,dx = 0$

미분 $\downarrow$ $\int \Big(\frac{\partial^2}{\partial\theta^2}\log f(x:\theta) + \Big(\frac{\partial}{\partial\theta}\log f(x:\theta)\Big)^2\Big)\exp(\log f(x:\theta))\,dx = 0.$

$\Leftrightarrow E\Big[\frac{\partial^2}{\partial\theta^2}\log f(x:\theta) + \Big(\frac{\partial}{\partial\theta}\log f(x:\theta)\Big)^2\Big] = 0$

$\Leftrightarrow \underbrace{E\Big[\Big(\frac{\partial}{\partial\theta}\log f(x:\theta)\Big)^2\Big]}_{=\,Var\left(\frac{\partial}{\partial\theta}\log f(x:\theta)\right)} = -E\Big[\frac{\partial^2}{\partial\theta^2}\log f(x:\theta)\Big].$

(4) 점근 정규성.

$\hookrightarrow |\theta^* - \theta| \le |\hat\theta - \theta|$.

(i) $0 = \dot{l}_n(\hat\theta) = \dot{l}_n(\theta) + \ddot{l}_n(\theta)(\hat\theta - \theta) + \frac{1}{2} l_n^{(3)}(\theta^*)(\hat\theta-\theta)^2$

(ii) $R_n = \frac{1}{2} l_n^{(3)}(\theta^*)(\hat\theta - \theta)$.

$\boxed{R6}$: $\hat\theta - \theta \xrightarrow{P} 0$, $\boxed{R7}$ $l_n^{(3)}(\theta^*) = O_p(1)$ $\Rightarrow R_n \xrightarrow{P} 0$.

(iii) $0 = \dot{l}_n(\theta) + (\hat\theta - \theta)\cdot\{\underbrace{\ddot{l}_n(\theta)}_{\downarrow P\ -I(\theta)} + \underbrace{R_n}_{\downarrow P\ 0}\}$

$\Rightarrow$ Under $\boxed{R0} - \boxed{R7}$, $\sqrt{n}(\hat\theta - \theta) \xrightarrow{d} N(0, (I(\theta))^{-1})$.

$* \ I(\theta) = E(-\frac{\partial^2}{\partial\theta^2}\log f(x;\theta)) = Var(\frac{\partial}{\partial\theta}\log f(x;\theta))$.

다차원 $\theta = (\theta_1 \dots \theta_k)^T$ $\left[-\frac{\partial^2}{\partial\theta\partial\theta^T}\log f(X;\theta)\right]_{k\times k}$.

6.4.6. One-Step Newton-Rhapson Estimator

Thm $\boxed{R0} \sim \boxed{R7} + \boxed{R8}$: $\sqrt{n}(\hat\theta_n^{(0)} - \theta)$가 극한분포를 갖는다. (i.e. $\hat\theta_n^{(0)}$ is $\sqrt{n}$-consistent est.) 준 MLE.

$\Rightarrow \hat\theta_n^{(1)} = \hat\theta_n^{(0)} + [-\ddot{l}(\hat\theta_n^{(0)})]^{-1} \dot{l}_n(\hat\theta_n^{(0)})$의 극한분포 $\leftarrow 0 = \dot{l}(\hat\theta^1) \simeq \dot{l}(\hat\theta^0) + \ddot{l}(\hat\theta^0)(\hat\theta^1 - \hat\theta^0)$.

: $\sqrt{n}(\hat\theta_n^{(1)} - \theta) \xrightarrow{d} N(0, (I(\theta))^{-1})$.

pf) Let $\hat\theta_n$ be the MLE. $\Rightarrow \dot{l}_n(\hat\theta_n) = 0$.

① $\tilde\theta_n$ 근방에서 $\dot{l}_n(\hat\theta_n)$ Taylor 전개. : $\exists\theta_n^*$ $\hookrightarrow |\theta_n^* - \hat\theta_n| \le |\tilde\theta_n - \hat\theta_n|$.

$0 = \dot{l}_n(\hat\theta_n) = \dot{l}_n(\tilde\theta_n) + \ddot{l}_n(\tilde\theta_n)(\hat\theta_n - \tilde\theta_n) + \frac{1}{2}\dddot{l}_n(\theta_n^*)(\hat\theta_n - \tilde\theta_n)^2$.

$\Leftrightarrow \dot{l}_n(\tilde\theta_n) = -\ddot{l}_n(\tilde\theta_n)(\hat\theta_n - \tilde\theta_n) - \frac{1}{2}\dddot{l}_n(\theta_n^*)(\hat\theta_n - \tilde\theta_n)^2$

② $\hat\theta_n^{(1)}$의 def : $0 = \dot{l}_n(\tilde\theta_n) + \ddot{l}_n(\tilde\theta_n)(\hat\theta_n^{(1)} - \tilde\theta_n)$.

$\Leftrightarrow \hat\theta_n^{(1)} = \tilde\theta_n + [-\ddot{l}_n(\tilde\theta_n)]^{-1}\dot{l}_n(\tilde\theta_n)$.

$\Leftrightarrow \hat\theta_n^{(1)} - \hat\theta_n = \tilde\theta_n - \hat\theta_n + [-\ddot{l}_n(\tilde\theta_n)]^{-1}\underline{\dot{l}_n(\tilde\theta_n)}$.

$\leftarrow$ ① plug-in.

$\Leftrightarrow \hat\theta_n^{(1)} - \hat\theta_n = (\tilde\theta_n - \hat\theta_n) + [-\ddot{l}_n(\tilde\theta_n)]^{-1}\{-\ddot{l}_n(\tilde\theta_n)(\hat\theta_n - \tilde\theta_n) - \frac{1}{2}\dddot{l}_n(\theta_n^*)(\hat\theta_n - \tilde\theta_n)^2\}$.

$= [-\ddot{l}_n(\tilde\theta_n)]^{-1} \times \frac{1}{2}\dddot{l}_n(\theta_n^*) \times (\hat\theta_n - \tilde\theta_n)^2$

$= I(\theta)^{-1}$ $\xleftarrow{P}$ (대수의 법칙). $\times$ // $\times$ //

$\left(\begin{array}{c}\sqrt{n}(\hat\theta_n - \tilde\theta_n) = \sqrt{n}(\hat\theta_n - \theta) - \sqrt{n}(\tilde\theta_n - \theta) = O_p(1) \\ \hat\theta_n - \tilde\theta_n = o_p(1)\end{array}\right)$

$\Leftrightarrow \sqrt{n}(\hat\theta_n^{(1)} - \hat\theta_n) = \underbrace{\{\frac{1}{2}\dddot{l}_n(\theta_n^*)\}}_{O_p(1)} \underbrace{\sqrt{n}(\hat\theta_n - \tilde\theta_n)}_{O_p(1)} \underbrace{(\hat\theta_n - \tilde\theta_n)}_{o_p(1)}$. $\xrightarrow{P} 0$.

Asymptotic Dist : $\sqrt{n}(\hat\theta_n^{(1)} - \theta_0) = \underbrace{\sqrt{n}(\hat\theta_n - \theta_0)}_{\xrightarrow{d} N(0, I(\theta)^{-1})} + \underbrace{\sqrt{n}(\hat\theta_n^{(1)} - \hat\theta_n)}_{\xrightarrow{P} 0} \xrightarrow{d} N(0, I(\theta)^{-1})$.

Ex ① $X_1 \ldots X_n \sim L(\theta, 1)$. $-\infty < \theta < \infty$.

$$f(x:\theta) = \frac{e^{x-\theta}}{(1+e^{x-\theta})^2} = \frac{e^{-x+\theta}}{(1+e^{-x+\theta})^2}$$

$$l_n(\theta) = \Sigma\{(\theta - x_i) - 2\log(1+e^{-x_i+\theta})\}$$

$$= n\theta - n\bar{x} - 2\log\Sigma(1+e^{-x_i+\theta}).$$

$$\Rightarrow \hat{\theta}^{(1)} = \hat{\theta}^{(0)} + [-\ddot{l}_n(\hat{\theta}^{(0)})]^{-1}\dot{l}_n(\hat{\theta}^{(0)})$$

Logistic 분포의 $E(X) = \theta \Rightarrow \hat{\theta}^{0} = \hat{\theta}^{MLE} = \bar{X}$.

$\hat{\theta}^{(1)} = \bar{X} + \left[2\Sigma \frac{e^{\bar{X}-x_i}}{(1+e^{\bar{X}-x_i})^2}\right]^{-1}\left[n - 2\Sigma\frac{e^{\bar{X}-x_i}}{1+e^{\bar{X}-x_i}}\right]$ 로 구할 수 있다.

$$\Rightarrow \sqrt{n}(\hat{\theta}^{(1)} - \theta) \xrightarrow{d} N(0, (I(\theta))^{-1}) = N(0, 3).$$

$$I(\theta) = E[-\ddot{l}(\theta)]$$

$$= 2E\left[\frac{e^{-x_i+\theta}}{(1+e^{-x_i+\theta})^2}\right] = 2E\left[\frac{U}{1-U}\right] = \frac{1}{3}.$$

$X_1 - \theta \overset{d}{=} \log\frac{U}{1-U}$, $\frac{1}{1+e^{-X+\theta}} \overset{d}{=} U \sim U(0,1)$.

Ex ② $X_1 \ldots X_n \sim Gamma(\alpha, \beta)$. $f(x:\theta) = \frac{1}{\Gamma(\alpha)\beta^\alpha}x^{\alpha-1}e^{-x/\beta}I(x>0)$. $\theta = (\alpha, \beta)^T$.

sol) $l_n(\theta) = -n\log\Gamma(\alpha) - n\alpha\log\beta + n(\alpha-1)\overline{\log X} - n\bar{X}/\beta$.

↳ $\dot{l}_n(\theta) = \begin{pmatrix} -n\Psi(\alpha) - n\log\beta + n\overline{\log X} \\ -n\alpha/\beta + n\bar{X}/\beta^2 \end{pmatrix}$

$\ddot{l}_n(\theta) = \begin{pmatrix} -n\dot{\Psi}(\alpha) & -n/\beta \\ -n/\beta & n\alpha/\beta^2 - 2n\bar{X}/\beta^2 \end{pmatrix}$

Let $\hat{\theta}_n^{(1)} = \hat{\theta}_n^{(0)} + [-\ddot{l}_n(\hat{\theta}_n^{(0)})]^{-1}\dot{l}_n(\hat{\theta}_n^{(0)})$

$\boxed{RS}$ $\sqrt{n}(\hat{\theta}_n^{(0)} - \theta_0)$ 가 극한분포 가진다.

$\hat{\alpha}_n\hat{\beta}_n = \bar{X}_n$

$\hat{\alpha}_n\hat{\beta}_n^2 = \frac{1}{n}\Sigma(X_i - \bar{X}_n)^2$.

$\Rightarrow I(\theta) = E(-\ddot{l}(\theta)) = \begin{pmatrix} \dot{\Psi}(\alpha) & 1/\beta \\ 1/\beta & \alpha/\beta^2 \end{pmatrix}$, $I(\theta)^{-1} = \frac{1}{\alpha\dot{\Psi}(\alpha) - 1}\begin{pmatrix} \alpha & -\beta \\ -\beta & \dot{\Psi}(\alpha)\beta^2 \end{pmatrix}$

$$\therefore \sqrt{n}(\hat{\theta}_n^{(1)} - \theta) \xrightarrow{d} N(0, (I(\theta))^{-1}).$$

③ 2차원 정규분포.

④ Cauchy 분포.

6.6. 추가 주제

6.6.1. Inconsistent MLE.

Ex $\begin{pmatrix} X_{11} & \cdots & X_{n1} \\ \vdots & \ddots & \vdots \\ X_{1k} & & X_{nk} \end{pmatrix} \to X_{ij} \overset{iid}{\sim} N(\mu_i, \sigma^2). \quad j=1, \dots, k.$

$$\hat{\sigma}^{2\,MLE} = \frac{1}{nk}\sum_{i=1}^{n}\sum_{j=1}^{k}(X_{ij}-\bar{X}_i)^2.$$

$$\frac{\Sigma(X_{ij}-\bar{X}_i)^2}{\sigma^2} \overset{iid}{\sim} \chi^2_{k-1}.$$

$$= \frac{1}{nk}\sum_{i=1}^{n}\left(\sum_{j=1}^{k}(X_{ij}-\bar{X}_i)^2\right) = \frac{1}{nk}\Sigma\sigma^2 W_i$$

$$= \frac{1}{n}\Sigma\frac{\sigma^2}{k}W_i$$

as $n\to\infty$, $\xrightarrow{P} \frac{\sigma^2}{k}E(W) = \frac{1}{k}(k-1)\sigma^2 \neq \sigma^2$

cf. Adjusted MLE $\frac{1}{n(k-1)}\Sigma\Sigma(X_{ij}-\bar{X}_i)^2$: consistent.

6.6.2. U-Statistics.

Def $U_n = \frac{1}{\binom{n}{r}}\sum_{1\le i_1<\cdots<i_r\le n} h(X_{i1}, \dots, X_{ir})$. : order r인 U-statistic.

$X_1 \dots X_n$ 中 랜덤하게 r개

EX ① (Kendall's τ) : $\hat{\tau}_n = \frac{1}{\binom{n}{2}}\sum_{1\le i<j\le n} \mathrm{sgn}((x_i-x_j)(y_i-y_j))$.

$= I((X_i-X_j)(Y_i-Y_j)>0)$.

$$= \frac{C_n - D_n}{\binom{n}{2}} \quad \begin{cases} C_n : \# \text{ of concordant pairs} \\ D_n : \# \text{ of non-concordant pairs.} \end{cases}$$

$$\Rightarrow \sqrt{n}(\hat{\tau}_n - \tau_n) \xrightarrow{d} N(0, \sigma^2).$$

② $s^2 = \frac{1}{n-1}\Sigma(X_i-\bar{X})^2 = U_n$.

$$= \frac{1}{\binom{n}{2}}\sum_{1\le i<j\le n}(X_i-X_j)^2\frac{1}{2}. \to \text{order } 2,\ h(x_1,x_2) = \frac{1}{2}(x_1-x_2)^2$$

③ $U_n = \frac{1}{\binom{n}{2}}\sum_{i<j}|x_i - x_j| \quad \left(\neq \frac{1}{n}\Sigma|X_i - \mathrm{median}(X_i)|.\right) \xrightarrow{P} E|X_1-X_2|$.

(unbiased).

④ $\frac{1}{n}\Sigma X_i^k$. ⑤ $\frac{1}{n}\Sigma I(X_i \le x_0)$ (fixed) $= F_n(x_0)$ ⑥ $\frac{1}{\binom{n}{2}}\sum_{i<j} I(X_i+X_j>0)$.

one-sample Wilcoxon stat.

→ symmetric test.

6.6.3. Asymptotic Dist. of U-Stat.

Def (Hájek Projection of U_n) $\hat{U}_n := \theta + \frac{r}{n}\Sigma h_1(X_i)$.

where $\theta = E[h(X_1 \dots X_n)]$, $h_1(X) = E[h(X_1 \dots X_r) | X_1 = x] - \theta$.

EX ② $\theta = E[\frac{1}{2}(X_1-X_2)^2] = \sigma^2$.

⑥ $\theta = E[I((X_1-X_2)(Y_1-Y_2)>0)] = P((X_1-X_2)(Y_1-Y_2)>0)$.

Thm (i) $U_n = \hat{U}_n + R_n$. $\sqrt{n} R_n \xrightarrow{P} 0$.

(ii) $\sqrt{n}(U_n - \theta) = \sqrt{n}(\hat{U}_n - \theta + U_n - \hat{U}_n) \xrightarrow{d} N(0, r^2 Var(h_1(X_1)))$.

$\xrightarrow{d} N(\ )$. $\xrightarrow{P} 0$.

Ex ② $h(x_1, x_2) = \frac{1}{2}(x_1 - x_2)^2$.

WLOG, $E(X_i) = 0$.

$h_1(x) = E[\frac{1}{2}(X_1 - X_2)^2 \mid X_1 = x] - \sigma^2$

$= E_{X_2}[\frac{1}{2}x^2 - xX_2 + \frac{1}{2}X_2^2] - \sigma^2 = \frac{1}{2}x^2 - \frac{1}{2}\sigma^2$.

$\Rightarrow \hat{U}_n = \sigma^2 + \frac{2}{n}\Sigma \frac{1}{2}(X_i^2 - \sigma^2)$

$= \frac{1}{n}\Sigma X_i^2$.

$\sqrt{n}(U_n - \hat{U}_n) = \sqrt{n} R_n = \sqrt{n}\left\{\frac{1}{n(n-1)}\Sigma X_i^2 - \frac{1}{n-1}(\sqrt{n}\bar{X})^2\right\} = o_p(1)$.

$U_n = \frac{1}{\binom{n}{2}}\Sigma \frac{1}{2}(X_i - X_j)^2 = \frac{1}{n-1}\Sigma(X_i - \bar{X})^2$.

③ $U_n = \frac{1}{\binom{n}{2}}\Sigma|X_i - X_j| \to h(x_1, x_2) = |x_1 - x_2|$. $\theta = E|X_1 - X_2|$. (WLOG $EX_i = 0$).

$h_1(x) = E[|X_1 - X_2| \mid X_1 = x] - \theta$.

↳ $X_i \overset{iid}{\sim} F$ 이면

$E|x - X| = \int_{-\infty}^{\infty}|x - t|\, dF(t)$.

$= \int_{-\infty}^{x}(x - t)\, dF(t) + \int_{x}^{\infty}(t - x)\, dF(t)$

$= (2F(x) - 1)\cdot x - 2\int_{-\infty}^{x} t\, dF(t) + \cancel{E(X)}^{0}$.

$\Leftrightarrow h_1(X_i) = (2F(x) - 1)X_i - 2\int_{-\infty}^{X_i} t\, dF(t) - \theta$.

$\Rightarrow \sqrt{n}(U_n - \theta) \xrightarrow{d} N(0, 4\sigma^2)$. $\sigma^2 = Var(h_1(X_i))$.

7장

7.1. 검정의 형식 논리.

7.1.1. 오류

$H_0 : \theta \in \Theta_0 \quad H_1 : \theta \in \Theta_1 . \qquad \Theta_0 \cap \Theta_1 = \phi.$

simple / composite.

- 제 1종 오류 : $P(\text{reject } H_0 \mid H_0 \text{ True})$.
- 제 2종 오류 : $P(\text{not reject } H_0 \mid H_1 \text{ True})$.

Ex $X_1 \dots X_{100} \overset{iid}{\sim} N(\mu, 5^2)$.

$H_0 : \mu = 30$ vs. $H_1 : \mu < 30$. $\{X : \bar{X} < c\} \rightarrow$ reject H_0.

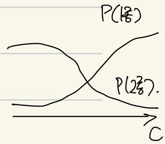
- $P(\text{Type 1 error}) = P(\mu = 30,\ \bar{X} < c)$

$$= P\left(\frac{\bar{X}-30}{5/\sqrt{100}} < \frac{c-30}{5/\sqrt{100}}\right) = \Phi(2(c-30)).$$

- $P(\text{Type 2 error}) = P(\mu = \mu' < 30,\ \bar{X} \geq c)$.

$$= P\left(\frac{\bar{X}-\mu'}{5/\sqrt{100}} \geq \frac{c-\mu'}{5/\sqrt{100}}\right) = 1 - \Phi(2(c-\mu')).$$

$\rightarrow$



7.1.2. 유의수준: 네이만-피어슨 생각.

(i) $P(\text{Type 1 error}) \leq \alpha. \quad \Leftrightarrow \quad E(\phi(X_1 \dots X_n)) \leq \alpha.$

(ii) $\max \{1 - P(\text{Type 2 error})\}$.

7.1.3. 용어들

① 기각역 : $R(X_1 \dots X_n) \ni (X_1 \dots X_n) \rightarrow$ Reject H_0.

② 기각 확률 함수 : $\phi(X_1 \dots X_n) = \begin{cases} 1 & \text{reject } H_0 \\ \gamma & \\ 0 & \text{accept } H_0. \end{cases}$

Ex $X_1 \dots X_n \overset{iid}{\sim}$ Poisson (θ)

$H_0 : \theta = 0.1$ vs. $H_1 : \theta < 0.1$. $\quad \alpha = 0.05$

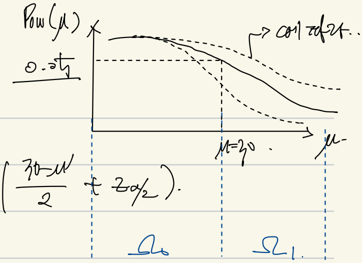
$\Rightarrow T = X_1 + \dots + X_{100} \sim$ Poisson $(100 \cdot \theta)$.

$P(T \leq c \mid H_0) \rightarrow$ $(c=4)$ 0.0292

$(c=5)$ 0.0670.

$\Rightarrow P(T=5)$ 이면 γ의 확률로 reject H_0.

$P(T \leq 4) + \gamma P(T=5) = 0.05$가 되는 γ 찾기.



7.1.4. 검정력 함수 (모든 θ에 대한 함수).

① 검정력 $= P(\text{reject } H_0)$.

$$= P(\mu=\mu',\ \bar{X} < c) \quad (c = 30+2\cdot z_{\alpha/2}) \quad = \Phi\left(\frac{30-\mu'}{2} + z_{\alpha/2}\right).$$

② Test 의 Unbiasedness.

$H_0: \theta \in \Omega_0$ vs. $H_1: \theta \in \Omega_1$.

$$\begin{cases} \max_{\theta\in\Omega_0} P(\text{type I error}) \le \alpha. \\ \min_{\theta\in\Omega_1} P(\text{reject } H_0) \ge \alpha. \end{cases}$$

7.2. 최대가능도비 검정.

7.2.1. LRT.

$X_1 \dots X_n$ iid $f(x;\theta)$. $H_0: \theta\in\Omega_0$ vs. $H_1: \theta\in\Omega_1$.

가능도함수 : $L(\theta) = \prod_{i=1}^{n} f(x_i;\theta)$.

① 검정통계량 : $\Lambda = \dfrac{\max_{\theta\in\Omega} L(\theta)}{\max_{\theta\in\Omega_0} L(\theta)} \ge c' \rightarrow$ reject H_0.

② 유의수준 α : $\sup_{\theta\in\Omega_0} P(\Lambda > c') \le \alpha$. 가 되도록 c'를 정한다.

$$\Leftrightarrow \quad 2\left(\ell(\hat\theta^{\Omega}) - \ell(\hat\theta^{\Omega_0})\right) > c''.$$

$\frac{1}{n}\sum(X_i-\hat\mu)^2$ 이므로 상수가 된다.

<u>Ex</u>. ① 정규분포 평균 / 양측. $H_0: \mu=\mu_0$ vs. $H_1: \mu\ne\mu_0$.

sol). $\theta = (\mu, \sigma^2)^T$, $\ell(\theta) = -\frac{1}{2\sigma^2}\sum(x_i-\mu)^2 - \frac{n}{2}\log\sigma^2 + $ 상수.

$$\hat\theta^{\Omega} = \left(\bar X,\ \frac{1}{n}\sum(X_i-\bar X)^2\right), \quad \hat\theta^{\Omega_0} = \left(\mu_0,\ \frac{1}{n}\sum(X_i-\mu_0)^2\right).$$

$$2\left(\ell(\hat\theta^{\Omega}) - \ell(\hat\theta^{\Omega_0})\right) = n\log\frac{\hat\sigma^2_{\Omega_0}}{\hat\sigma^2_{\Omega}}$$

$$= n\log\frac{\sum(X_i-\mu_0)^2}{\sum(X_i-\bar X)^2}$$

$$= n\log\left(1+\frac{(\bar X-\mu_0)^2}{\underset{\displaystyle \frac{(n-1)s^2}{n}}{\underset{\parallel}{\hat\sigma^2_{\Omega}}}}\right) \ge c.$$

$$\Leftrightarrow \quad \underbrace{\left|\frac{\bar X-\mu_0}{s/\sqrt{n}}\right|}_{\sim\ t(n-1)} \ge c.$$

$t(n-1)$ 이므로 기각역 : $\left|\dfrac{\bar X-\mu_0}{s/\sqrt{n}}\right| \ge t_{\alpha/2}(n-1)$..

$-\frac{1}{2\sigma^2}\Sigma(X_i-\hat\mu)^2$.
$\Sigma(X_i-\bar X)^2 + n(\bar X-\mu)^2 \downarrow$.
$\mu \le \mu_0$.

② 정규분포 평균 / 단측. $H_0: \mu \le \mu_0$ vs. $H_1: \mu > \mu_0$.

$\bar X$. μ_0. $\bar X$
$\hat\mu = \bar X$. $\hat\mu = \mu_0$

sol) $\hat\theta^{\Omega} = (\bar X, \frac{1}{n}\Sigma(X_i-\bar X)^2)$.

$$\hat\mu^0 = \min(\bar X, \mu_0), \quad \hat\sigma^{2\,0} = \frac{1}{n}\Sigma(X_i-\hat\mu^0)^2.$$

$$2\{l(\hat\theta^\Omega) - l(\hat\theta^{\Omega_0})\} = n\log\frac{\hat\sigma^{2\,\Omega_0}}{\hat\sigma^{2\,\Omega}}$$

$$= \begin{cases} 0 & \bar X \le \mu_0. \\ n\log\dfrac{\Sigma(X_i-\mu_0)^2}{\Sigma(X_i-\bar X)^2} & \bar X > \mu_0. \end{cases} \triangleq T.$$

기각역: $\sup_{\mu\le\mu_0} P(T>c) \le \alpha$.

$\mu=\mu_0$인 경우 $T>c \iff \dfrac{\Sigma(X_i-\bar X)^2 + n(\bar X-\mu_0)^2}{\Sigma(X_i-\bar X)^2} > c$.

$$\iff \frac{\bar X-\mu_0}{\sqrt{\frac{1}{n}S_n^2}} > c'. \quad \to c' = t_\alpha(n-1).$$

$$\sup_{\mu\le\mu} P(T>c)\le\alpha \iff \sup P\left(\frac{\bar X-\mu}{\sqrt{\frac{1}{n}s^2}} + \frac{\mu-\mu_0}{\sqrt{\frac{1}{n}s^2}} > c_2\right) \le \alpha.$$

$$\iff \sup P\left(T > c - \frac{\mu-\mu_0}{\sqrt{\frac{1}{n}s^2}}\right) \le \alpha$$

$= c_2 \ge 0$ 이므로 $\frac{\mu-\mu_0}{\sqrt{\frac{1}{n}s^2}} = 0$일 때 최대.

$T \sim t(n-1)$ 이므로 $c_2 = t_\alpha(n-1)$.

$$\therefore T = \frac{\bar X-\mu_0}{\sqrt{\frac{1}{n}s^2}} > t_\alpha(n-1) \Rightarrow \text{reject } H_0.$$

③ 지수분포. 평균.

$f(x;\theta) = \frac{1}{\theta}\exp(-\frac{1}{\theta}x)$. $H_0: \theta = \theta_0$ vs. $H_1: \theta \ne \theta_0$.

sol). $l(\theta) = -n\log\theta - \Sigma\frac{n}{\theta}\bar X$.

$\hat\theta^\Omega = \bar X$, $\hat\theta^0 = \theta_0$.

$$2\{l(\hat\theta^\Omega) - l(\hat\theta^0)\} = 2\left(n\log\theta_0 + \frac{n}{\theta_0}\bar X - n\log\bar X - n\right).$$

$$= 2n\left(\frac{\bar X}{\theta_0} - \log\frac{\bar X}{\theta_0} - 1\right).$$

기각역: // $> c$.

$\iff \frac{\bar X}{\theta_0} - \log\frac{\bar X}{\theta_0} - 1 > c_1$. $\to$ let $h(t) = t - \log t - 1$.

$$\iff \exists d_1, d_2, \text{ s.t. } \begin{cases} \frac{\bar X}{\theta_0} < d_1, \ \frac{\bar X}{\theta_0} > d_2. & \cdots ① \\ h(d_1) = h(d_2). & \cdots ② \end{cases}$$

② → $d_1 - \log d_1 = d_2 - \log d_2$.

① → $P\left(d_1 < \frac{\bar{X}}{\theta_0} < d_2\right) = \alpha$. $\frac{n\bar{X}}{\theta_0} \sim Gamma(n, \frac{1}{\theta_0})$ 이므로

$\longrightarrow \int_{nd_1}^{nd_2} pdf_{Gamma(n, \frac{1}{\theta_0})}(t)\, dt = 1-\alpha$

④ 정규분포, 평균 비교. $H_0 : \mu_1 = \mu_2$ vs. $H_1 : \mu_1 \neq \mu_2$.

$X_{11} \dots X_{1n_1} \overset{iid}{\sim} N(\mu_1, \sigma^2)$, $X_{21} \dots X_{2n_2} \overset{iid}{\sim} N(\mu_2, \sigma^2)$.

sol) $l(\mu_1, \mu_2, \sigma^2) = -\frac{1}{2\sigma^2} \sum\sum (x_{ij} - \mu_i)^2 - \frac{n_1+n_2}{2} \log \sigma^2 + C$.

$\hat{\theta}^{\Omega} = (\hat{\mu}_1, \hat{\mu}_2, \hat{\sigma}_1^2) = \left(\bar{x}_1, \bar{x}_2, \frac{1}{n_1+n_2} \sum (X_{ij} - \bar{X}_i)^2\right)$.

$\hat{\theta}^{0} = (\hat{\mu}^0, \hat{\mu}^0, \hat{\sigma}^{2}_0)$, $\begin{cases} \hat{\mu}^0 = \frac{n_1\bar{X}_1 + n_2\bar{X}_2}{n_1+n_2} \\ \hat{\sigma}^{2\,0} = \frac{1}{n_1+n_2} \sum\sum (X_{ij} - \hat{\mu}^0)^2. \end{cases}$

$\Rightarrow 2\left(l(\hat{\theta}^{\Omega}) - l(\hat{\theta}^{0})\right) = (n_1+n_2) \log \frac{\hat{\sigma}^{2}_0}{\hat{\sigma}^2}$ $\quad \sum\sum (X_{ij} - \bar{X}_i)^2 + (n_1+n_2) \sum (\bar{X}_i - \hat{\mu}_0)^2$

$= (n_1+n_2) \log \frac{\sum\sum (X_{ij} - \hat{\mu}^0)^2}{\sum\sum (X_{ij} - \bar{X}_i)^2}$.

$\overset{+}{\sim} \quad (\bar{X}_1 - \hat{\mu}_0)^2 + (\bar{X}_2 - \hat{\mu}_0)^2$

$= \frac{(\bar{X}_1 - \bar{X}_2)^2 (n_2^2 + n_1^2)}{(n_1+n_2)^2}$

$\overset{+}{\sim} \quad \frac{\bar{X}_1 - \bar{X}_2}{\sqrt{\frac{1}{n_1} + \frac{1}{n_2}} \cdot S_p} \sim t(n_1+n_2-2)$

$\therefore \left| \frac{\bar{X}_1 - \bar{X}_2}{\sqrt{\frac{1}{n_1}+\frac{1}{n_2}} S_p} \right| \geq t_{\alpha/2}(n_1+n_2-2) \Rightarrow$ reject H_0.

⑤ 정규분포, 분산 비교 $H_0 : \sigma_1^2 = \sigma_2^2$ vs. $H_1 : \sigma_1^2 \neq \sigma_2^2$.

$X_{11} \dots X_{1n_1} \overset{iid}{\sim} N(\mu_1, \sigma_1^2)$, $X_{21} \dots X_{2n_2} \overset{iid}{\sim} N(\mu_2, \sigma_2^2)$.

sol) $l(\mu_1, \mu_2, \sigma_1^2, \sigma_2^2) = -\frac{n_1}{2} \log \sigma_1^2 - \frac{1}{2\sigma_1^2} \sum (X_{1j} - \mu_1)^2$
$\qquad - \frac{n_2}{2} \log \sigma_2^2 - \frac{1}{2\sigma_2^2} \sum (X_{2j} - \mu_2)^2$.

$\hat{\theta}^{\Omega} = \Big(\bar{x}_1, \bar{x}_2, \underbrace{\frac{1}{n_1} \sum (X_{1j} - \bar{X}_1)^2}_{= \hat{\sigma}_1^2}, \underbrace{\frac{1}{n_2} \sum (X_{2j} - \bar{X}_2)^2}_{= \hat{\sigma}_2^2}\Big)$

$\hat{\theta}^{\Omega_0} = \Big(\bar{X}_1, \bar{X}_2, \underbrace{\frac{1}{n_1+n_2} \left\{\sum (X_{1j} - \bar{X}_1)^2 + \sum (X_{2j} - \bar{X}_2)^2\right\}}_{= \hat{\sigma}^2}\Big)$.

$2\left(l(\hat{\theta}) - l(\hat{\theta}_0)\right) = (n_1+n_2) \log \hat{\sigma}^2 - n_1 \log \hat{\sigma}_1^2 - n_2 \log \hat{\sigma}_2^2 \longrightarrow$ ⊛

$\hat\sigma^2 = \dfrac{n_1\hat\sigma_1^2 + n_2\hat\sigma_2^2}{n_1+n_2}$ 이므로,

$$⊛ = -n_1 \log\frac{\hat\sigma_1^2}{\hat\sigma^2} - n_2\log\frac{\hat\sigma_2^2}{\hat\sigma^2}, \quad t = \frac{n_1\hat\sigma_1^2}{n_1\hat\sigma_1^2+n_2\hat\sigma_2^2}$$ 로 두면.

기각역: $-n_1\log t - n_2\log(1-t) > c_1$.

$h(t) = -n_1\log t - n_2\log(1-t)$ 로 두면

$$\Leftrightarrow \begin{cases} t<c_1,\ t>c_2. & —① \\ n_1\log c_1 + n_2\log(1-c_1) = n_1\log c_2 + n_2\log(1-c_2). & —② \end{cases}$$

① $\Leftrightarrow$ $P_{\sigma_1^2=\sigma_2^2}(c_1<t<c_2)=1-\alpha$ 이고,

$\hat\sigma_1^2 = \dfrac{n_1-1}{n_1}S_1^2$, $\hat\sigma_2^2=\dfrac{n_2-1}{n_2}S_2^2$ 를 이용하면

$$t = \frac{(n_1-1)S_1^2}{(n_1-1)S_1^2+(n_2-1)S_2^2} = \frac{n_1-1}{(n_1-1)+(n_2-1)\frac{S_2^2}{S_1^2}}.$$

$$\sim \frac{S_1^2}{S_2^2} = F.$$

① $\Leftrightarrow$ $P(d_1<F<d_2)=1-\alpha$, $c_1 = \dfrac{(n_1-1)d_1}{(n_1-1)d_1+(n_2-1)}$. $c_2 = \dfrac{(n_1-1)d_2}{(n_1-1)d_2+(n_2-1)}$

$\therefore$ $F\le d_1$, $F\ge d_2$ $\to$ reject H_0.

⑥ 포아송분포, 단측.

$X_1 \dots X_{100} \sim Poi(\theta)$. $\theta>0$. $H_0: \theta\ge 0.1$ vs. $H_1: \theta<0.1$.

sol. $\ell(\theta) = -n\theta + n\bar x\log\theta +$ 상수

$\hat\theta^\Omega = \bar x$, $\hat\theta^0 = \max(\bar x, 0.1)$.

$$2(\ell(\hat\theta^\Omega)-\ell(\hat\theta^0)) = 2(-n\bar x + n\bar x\log\bar x + n\cdot 0.1 - n\bar x\log 0.1)I(\bar x\le 0.1).$$
$$= 2n(\bar x\log\bar x - \bar x(1+\log 0.1) + 0.1)I(\bar x\le 0.1).$$

↓ $\bar x$의 감소함수.

$\Rightarrow$ 기각역: $\bar X\le c$. $\max_{\theta\ge 0.1} P_\theta(X_1+\cdots+X_n\le k) = P_{\theta=0.1}(X_1+\cdots+X_n\le k)$.
↳ 감소함수

$$= \max_{\theta\ge 0.1} E[\phi(X_1\cdots X_{100})].$$

$$\phi(X_1\cdots X_{100}) = \begin{cases} 1 & \text{if } x_1+\cdots+x_{100}\le 4 \\ \gamma & =5 \\ 0 & \ge 6. \end{cases} \to P(X_1+\cdots+X_{100}\le 4) + \gamma P(X_1+\cdots+X_{100}=5)$$
$$= 0.05. \qquad \gamma = \frac{21}{38}.$$

7.3. 최대가능도 검정법의 근사.

7.3.2. 단순귀무가설의 경우.

Thm. $H_0 : \theta = \theta_0$ vs. $H_1 : \theta \neq \theta_0$. (R0-R7)

(a) $2\{ l(\hat\theta) - l(\theta_0) \} \xrightarrow{d} \chi^2(k)$.

(b) $2\{ l(\hat\theta) - l(\theta_0) \} = \sqrt{n}(\hat\theta_n - \theta_0)^T I(\theta_0) \sqrt{n}(\hat\theta_n - \theta_0) + r_n$. $r_n \xrightarrow{P} 0$.

$\rightarrow W_n = n(\hat\theta_n - \theta_0)^T I(\theta_0)(\hat\theta_n - \theta_0) \xrightarrow{d} \chi^2(k)$ $\bar{\dot l}_n(\theta) = \frac{1}{n}\frac{\partial}{\partial\theta} l(\theta)$.

(c) $2\{ l(\hat\theta) - l(\theta_0) \} = n\, \bar{\dot l}_n(\theta_0)^T [I(\theta_0)]^{-1} \bar{\dot l}_n(\theta_0) + r_n$. $r_n \xrightarrow{P} 0$.

$\rightarrow R_n(\theta_0) = n\, \bar{\dot l}_n(\theta_0)^T [I(\theta_0)]^{-1} \bar{\dot l}_n(\theta_0) \xrightarrow{d} \chi^2(k)$.

pf) (c): $0 = \bar{\dot l}_n(\hat\theta_n) = \bar{\dot l}_n(\theta_0) + \bar{\ddot l}_n(\theta_0^*)(\hat\theta_n - \theta_0)$ $\exists\, \theta_0^*$ s.t. $|\theta_0^* - \theta_0| \le |\hat\theta_n - \theta_0|$.

$= -\bar{\ddot l}_n(\theta_0^*)(\hat\theta_n - \theta_0)$. L.L.N.

$\simeq I(\theta_0)(\hat\theta_n - \theta_0)$

Ex ① 지수분포, 평균. $H_0 : \theta = \theta_0$ vs. $H_1 : \theta \neq \theta_0$.

$\bar{\ddot l}(\theta) = \frac{1}{\theta^2} - \frac{2\bar X}{\theta^3}$.

$E[-\bar{\ddot l}(\theta)] = -\frac{1}{\theta^2} + \frac{2\theta}{\theta^3} = \frac{1}{\theta^2}$.

sol). $l(\theta) = -n\log\theta - n\bar X/\theta$. $\bar{\dot l}(\theta) = -\frac{1}{\theta} + \frac{\bar X}{\theta^2}$. $I(\theta) = \frac{1}{\theta^2}$. $\hat\theta^{\Omega} = \bar X$.

(a) LRT: $2\{ l(\hat\theta) - l(\theta_0) \} \overset{mt}{=} 2n\left\{ \frac{\bar X}{\theta_0} - 1 - \log\frac{\bar X}{\theta_0} \right\}$.

$\left(\begin{array}{l} h(t) = t - 1 - \log t \rightarrow \text{1에서 taylor전개} \\ h\left(\frac{\bar X}{\theta_0}\right) \simeq \left(\frac{\bar X - \theta_0}{\theta_0}\right)^2 \cdot \frac{1}{2} \end{array} \right)$ $\hookrightarrow \frac{n(\bar X - \theta_0)^2}{\theta_0^2} \xrightarrow{d} \chi^2(1)$.

(b) Wald, Rao $\Rightarrow$ By def. $\frac{n(\bar X - \theta_0)^2}{\theta_0^2} \xrightarrow{d} \chi^2(1)$.

② 다항분포. p: $p_1 + \cdots + p_k = 1$, $p_i > 0$.

$Z_1 \dots Z_n \overset{iid}{\sim} \text{Multi}(1, p = (p_1 \dots p_k)^T)$. $H_0 : p = p_0$ vs. $H_1 : p \neq p_0$

sol). Let $X = (X_1 \dots X_k)$. $X_i = \sum_{j=1}^{n} Z_{ij}$. $\sim \text{Multi}(n, p)$

$\left< \begin{array}{l} \theta = (\theta_1 \dots \theta_r), \quad \theta_i = p_i, \quad i = 1 \dots r = k-1 \\ \theta_\cdot = \theta_1 + \cdots + \theta_r = 1 - p_k \end{array} \right.$

$l(\theta) = \sum^{r} x_i \log\theta_i + x_k \log(1 - \theta_\cdot) + C$.

$$\dot l(\theta) = \begin{bmatrix} \frac{x_1}{\theta_1} - \frac{x_k}{1-\theta_\cdot} \\ \vdots \\ \frac{x_r}{\theta_r} - \frac{x_k}{1-\theta_\cdot} \end{bmatrix} \rightarrow \hat\theta^{\Omega} = \left(\frac{x_1}{n} \dots \frac{x_r}{n}\right)^T.$$

$-\ddot l(\theta) = \text{diag}\left(\frac{x_i}{\theta_i^2}\right) + \frac{x_k}{(1-\theta_\cdot)^2} 1_r 1_r^T$

$I(\theta) = \text{diag}\left(\frac{1}{\theta_i}\right) + \frac{1}{(1-\theta_\cdot)} 1_r 1_r^T$.

(a) LRT: $2\{l(\hat\theta^{\Omega}) - l(\theta_0)\} = 2\left\{\sum x_i \log \frac{x_i}{np_{0i}} + x_k \log \frac{x_k}{np_{0k}}\right\}$.

$$= 2\left\{\sum_{i=1}^{k} x_i \log \frac{x_i}{np_{0i}}\right\}.$$

(b) $W_n(\theta_0) = n(\hat\theta_n^{\Omega} - \theta_0)^T I(\theta_0)(\hat\theta_n^{\Omega} - \theta_0)$

$$= n\left[\frac{x_1}{n} - \theta_{01} \;\cdots\; \frac{x_r}{n} - \theta_{0r}\right] \times \left\{\begin{bmatrix} \frac{1}{\theta_1} & & \\ & \ddots & \\ & & \frac{1}{\theta_r} \end{bmatrix} + \frac{1}{1-\theta_{0\cdot}} 1_r 1_r^T\right\} \times \left[\;\;''\;\;\right]$$

$$= \sum^{r} \frac{(x_i - n\theta_{0i})^2}{n\theta_{0i}} + \frac{1}{1-\theta_0}\left(\frac{x_1 + \cdots + x_r}{n} - \theta_0\right)^2$$

$$= \frac{1}{p_{0k}}\left(\frac{n - x_k}{n} - (1 - p_{0k})\right)^2 = \frac{(x_k - np_{0k})^2}{n\,p_{0k}}$$

$$= \sum^{k} \frac{(x_i - np_{0i})^2}{np_{0i}} \xrightarrow{d} \chi^2(k-1)$$

7.3.4. 일반적인 귀무가설의 경우.

<u>Thm</u> $X_1 \cdots X_n \overset{iid}{\sim} f(x;\theta)$. $\theta = (\xi^T, \eta^T)^T \in \mathbb{R}^k$.

$H_0: \xi = \xi_0$ vs. $H_1: \xi \neq \xi_0$ $\rightarrow$ $\Omega_0 = \{(\xi_0^T, \eta^T)^T : \eta \in \mathbb{R}^{k_0}\}$.

① $2\{l(\hat\theta^{\Omega}) - l(\hat\theta^{0})\} \xrightarrow{d} \chi^2(k - k_0)$.

② $2\{l(\hat\theta^{\Omega}) - l(\hat\theta^{0})\} = n(\hat\theta^{\Omega} - \hat\theta^{\Omega_0})^T I(\theta_0)(\hat\theta^{\Omega} - \hat\theta^{\Omega_0}) + r_n$. $r_n \xrightarrow{P} 0$.

③ $2\{l(\hat\theta^{\Omega}) - l(\hat\theta^{0})\} = n\,\bar{l}_n(\hat\theta^{\Omega_0})^T [I(\theta_0)]^{-1} \bar{l}_n(\hat\theta^{\Omega_0}) + r_n$. $r_n \xrightarrow{P} 0$.

<u>Ex</u>. ① $r \times c$ 분할표, 독립성 검정.

$(X_{ij}) \sim \text{Multi}(n, (p_{ij}))$, $\sum_{i=1}^{r}\sum_{j=1}^{c} p_{ij} = 1$, $p_{ij} > 0$.

$H_0: p_{ij} = p_{i\cdot} \times p_{\cdot j}$ vs. H_1: not H_0.

Sol) $l(p) = \sum\sum x_{ij} \log p_{ij} + (\text{상수})$. $p_{ij} = p_{i\cdot} \times p_{\cdot j}$.

Under H_0, $l(p^0) = \sum x_{i\cdot} \log p_{i\cdot}^0 + \sum x_{\cdot j} \log p_{\cdot j}^0 + (\text{상수})$

(a) $2\{l(\hat p) - l(\hat p^0)\} = 2\sum\sum x_{ij} \log \frac{\hat p_{ij}}{\hat p_{ij}^0} \xrightarrow{d} \chi^2(k - k_0)$.

$\sum\sum p_{ij} = 1$.
$k = rc - 1$
$k_0 = (r-1) + (c-1)$
$k - k_0 = (r-1)(c-1)$.
$\sum p_{i\cdot} = 1$, $\sum p_{\cdot j} = 1$.

$\hookrightarrow$ $\hat p_{ij} = \frac{x_{ij}}{n}$, $\hat p_{ij}^0 = \hat p_{i\cdot}^0 \times \hat p_{\cdot j}^0$, $\hat p_{i\cdot} = \frac{X_{i\cdot}}{n}$, $\hat p_{\cdot j} = \frac{X_{\cdot j}}{n}$.

$H_0: p_{ij} = p_{i\cdot}^0 \times p_{\cdot j}^0$ (독립 가정)

(b) $W_n(\hat p^0) = R_n(\hat p^0) = \sum\sum (X_{ij} - n\hat p_{i\cdot}^0 \hat p_{\cdot j}^0)^2 / (n\hat p_{i\cdot}^0 \hat p_{\cdot j}^0)$.

$$\xrightarrow{d} \chi^2(k - k_0) = \chi^2((r-1)(c-1)).$$

② 독립인 다항분포. 동일성 검정

$X_i = (X_{i1} \dots X_{ic})^T \sim \text{Multi}(n_i, p_i = (p_{i1} \dots p_{ic})^T)$, $i = 1, \dots, r$.

$X_1 \dots X_r$ 은 서로 독립.

$H_0 : p_1 = \cdots = p_r$ vs. H_1 : not H_0.

sol) $l(p) = \sum\sum x_{ij} \log p_{ij} + \cancel{\text{c}}$.

$\Rightarrow \hat{p}_i = \dfrac{x_i}{n_i}$, $\hat{p}_1^0 = \cdots = \hat{p}_r^0 = x_\cdot / n = (x_1 + \cdots + x_r)/(n_1 + \cdots + n_r)$.

(a) $$2\{ l(\hat{p}) - l(\hat{p}^0) \} = 2 \sum\sum x_{ij} \log \frac{\hat{p}_{ij}}{\hat{p}_{ij}^0}$$

$$= 2 \sum\sum n_i \hat{p}_{ij} \log \frac{\hat{p}_{ij}}{\hat{p}_{ij}^0} = 2\sum\sum n_i \hat{p}_{ij}^0 \frac{\hat{p}_{ij}}{\hat{p}_{ij}^0} \log \frac{\hat{p}_{ij}}{\hat{p}_{ij}^0}$$

$$\simeq 2\sum\sum n_i \hat{p}_{ij}^0 \left\{ \left(\frac{\hat{p}_{ij}}{\hat{p}_{ij}^0} - 1\right) + \frac{1}{2}\left(\frac{\hat{p}_{ij}}{\hat{p}_{ij}^0} - 1\right)^2 \right\}$$

$$= \sum\sum n_i \hat{p}_{ij}^0 \left(\frac{\hat{p}_{ij}^2}{\hat{p}_{ij}^{0\,2}} - \hat{p}_{ij}^{0\,2} \right)$$

$$= \sum\sum \frac{(O_{ij} - \hat{E}_{ij}^0)^2}{\hat{E}_{ij}^0}$$

Under $H_0 : p_1 = \cdots = p_r = p^0$, $\Rightarrow \sum\sum n_i (\hat{p}_{ij} - \hat{p}_{ij}^0)^2 / \hat{p}_{ij}^0$

$\hat{p}_i^0 \simeq p^0$. $\simeq \sum\sum n_i (\hat{p}_{ij} - \hat{p}_{ij}^0)^2 / p_j^0$.

$= \sum n_i (\hat{\theta}_i - \hat{\theta}_i^0)^T \Sigma_0^{-1} (\hat{\theta}_i - \hat{\theta}_i^0)$

($\hat{\theta}_i, \hat{\theta}_i^0$ 은 $\hat{p}_i, \hat{p}_i^0$ 의 첫 $(c-1)$ 성분으로 구성된 벡터)

$\Sigma_0 = \text{diag}(\theta_j^0) - \theta^0 \theta^{0T}$.

Let $Z_i = \Sigma_0^{-1/2} \sqrt{n_i} (\hat{\theta}_i - \theta^0) \to Z_i \sim N(0, I)$.

$\hat{\theta}_i^0 - \theta^0 = \sum n_l (\hat{\theta}_l - \theta^0) / n = \sum \sqrt{n_l}\, \Sigma_0^{1/2} Z_l / n$.

$\Rightarrow \Sigma_0^{-1/2} (\hat{\theta}_i^0 - \theta^0) = \sum w_l Z_l / w^T w$. $w_l = \sqrt{n_l}$.

$$\Rightarrow \sum n_i (\hat{\theta}_i - \hat{\theta}^0) \Sigma_0^{-1} (\hat{\theta}_i - \hat{\theta}^0) = \begin{bmatrix} Z_1^T \\ \vdots \\ Z_r^T \end{bmatrix} \left(I_r \otimes I_d - w(w^T w)^{-1} w^T \otimes I_d \right) [Z_1^T \dots Z_r^T]^T$$

$$\xrightarrow{d} \chi^2(rd - d) = \chi^2((r-1)(c-1)).$$

(b) $W_n(\hat{p}^0) = R_n(\hat{p}^0) = \sum\sum (X_{ij} - n_i \hat{p}_{ij}^0)^2 / (n_i \hat{p}_{ij}^0)$.

$$\xrightarrow{d} \chi^2((r-1)(c-1)).$$

7.4. 추가문제

7.4.1. Irregular Problems.

Ex ① Constraints 가 있는 문제

$X_1 \dots X_n \overset{iid}{\sim} N_2\left(\begin{pmatrix}\theta_1\\ \theta_2\end{pmatrix}, I\right)$, $X_i = \begin{pmatrix}X_{i1}\\ X_{i2}\end{pmatrix}$ 독립.

$H_0: \theta_1=\theta_2=0$. $H_1: \theta_1>0,\ \theta_2>0$.

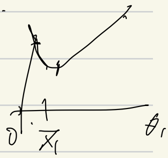
$\Rightarrow \Omega_0 = \{(\theta_1,\theta_2)^T : \theta_1=\theta_2=0\}$. $\Omega = \{(\theta_1,\theta_2)^T = \theta_1\ge 0,\ \theta_2\ge 0\}$.

sol) Let $\theta = (\theta_1, \theta_2)^T$.

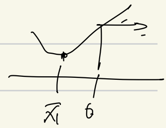
$\hat\theta^{\Omega_0} = (0,0)^T$ $\quad \hat\theta^{\Omega} = \left(\max(\bar X_1, 0),\ \max(\bar X_2, 0)\right)$.

$-\frac{1}{2\sigma^2}\sum\sum(X_{ij}-\theta_i)^2$

↳ $\sum(X_{1j}-\bar X_1)^2 + n(\bar X_1-\theta_1)^2$
$+\sum(X_{2j}-\bar X_2)^2 + n(\bar X_2-\theta_2)^2$.



검정통계량 $\Lambda_n = \dfrac{\max_{\theta\in\Omega_0} L(\theta)}{\max_{\theta\in\Omega} L(\theta)} = \dfrac{\exp\{-\frac12\sum(X_{i1}^2+X_{i2}^2)\}}{\exp\{-\frac12\{\sum(X_{i1}-\hat\theta_1)^2+\sum(X_{i2}-\hat\theta_2)^2\}\}}$.



(i) $\bar X_1\le 0$, $\bar X_2\le 0$. $\rightarrow \Lambda_n = 0$

(ii) $\bar X_1>0$, $\bar X_2\le 0$ $\rightarrow \Lambda_n = \exp\{-\frac12\sum X_{i1}^2\} / \exp\{-\frac12\sum(X_{i1}-\bar X_1)^2\}$.

(iii) $\bar X_1\le 0$, $\bar X_2>0$. $\rightarrow \Lambda_n = \exp\{-\frac12\sum X_{i2}^2\} / \exp\{-\frac12\sum(X_{i2}-\bar X_2)^2\}$.

(iv) $\bar X_1>0$, $\bar X_2>0$. $\rightarrow \Lambda_n = \exp\{-\frac12\sum(X_{i1}^2+X_{i2}^2)\} / \exp\{-\frac12\{\sum(X_{i1}-\bar X_1)^2+\sum(X_{i2}-\bar X_2)^2\}\}$.

Under $H_0: \theta_1=\theta_2=0$,

$P(2\log\Lambda_n \le t) = P((i), 2\log\Lambda_n\le t) + P((ii), 2\log\Lambda_n \le t) +$
$+ P((iii), 2\log\Lambda_n\le t) + P((iv), 2\log\Lambda_n\le t)$

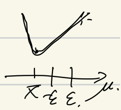
p0 wp1. $\quad \xrightarrow{d} \chi^2(1)$. $\quad \xrightarrow{d} \chi^2(1)$. $\quad \xrightarrow{d} \chi^2(2)$.

$\xrightarrow{d} \frac14\delta_{(0)} + \frac12\chi^2(1) + \frac14\chi^2(2)$

② Bioequivalence Test.

$X_1\dots X_n \overset{iid}{\sim} N(\mu,\sigma^2)$. $H_0: |\mu|\le\varepsilon$ vs. $H_1: |\mu|>\varepsilon$.

sol) $\hat\mu^{\Omega} = \bar X$, $\hat\mu^{\Omega_0} = \begin{cases}\bar X & |\bar X|\le\varepsilon\\ -\varepsilon & \bar X<-\varepsilon\\ \varepsilon & \bar X>\varepsilon.\end{cases}$ $\quad -\frac{1}{2\sigma^2}\sum(X_i-\hat\mu)^2$



$-2\log\Lambda_n = \begin{cases}0 & |\bar X|\le\varepsilon\\ \frac{n}{\sigma^2}(\bar X+\varepsilon)^2 & \bar X<-\varepsilon\\ \frac{n}{\sigma^2}(\bar X-\varepsilon)^2 & \bar X>\varepsilon.\end{cases}$

$\rightarrow \frac{n}{\sigma^2}(\bar X-\mu)^2 + \frac{n}{\sigma^2}(\mu-\varepsilon)^2$
$\xrightarrow{d}$ NC $\chi^2(1, (\mu-\varepsilon)^2/\sigma^2)$.

$$\begin{cases} P(|\bar{X}| \le \varepsilon) = \Phi\left(\frac{\sqrt{n}(\varepsilon-\mu)}{\sigma}\right) + \Phi\left(\frac{\sqrt{n}(\varepsilon+\mu)}{\sigma}\right) - 1. \\ P(\bar{X} < -\varepsilon) = \Phi\left(-\frac{\sqrt{n}(\varepsilon+\mu)}{\sigma}\right). \\ P(\bar{X} > \varepsilon) = 1 - \Phi\left(\frac{\sqrt{n}(\varepsilon-\mu)}{\sigma}\right) \end{cases}$$

$\Rightarrow$ (i) $|\mu| \le \varepsilon$ then $P(|\bar{X}| \le \varepsilon) \to 1.$ $\quad -2\log\Lambda_n \xrightarrow{d} \delta_{(0)}$

(ii) $\mu = -\varepsilon$ then $P(\bar{X} < -\varepsilon), P(|\bar{X}| \le \varepsilon) \to 1/2.$ $\quad -2\log\Lambda_n \xrightarrow{d} \frac{1}{2}\delta_{(0)} + \frac{1}{2}\chi^2(1).$

$\mu = \varepsilon$ then $P(\bar{X} > \varepsilon)$, 〃 $\to 1/2.$ $\quad -2\log\Lambda_n \xrightarrow{d}$ 〃

(iii) $\mu > \varepsilon$ then $P(|\bar{X}| \le \varepsilon) \to 0$, $P(\bar{X} > \varepsilon) \to 1.$ $\quad -2\log\Lambda_n \xrightarrow{d} NC\,\chi^2(1, (\mu-\varepsilon)^2/\sigma^2)$

(iv) $\mu < -\varepsilon$ Then 〃 $P(\bar{X} < \varepsilon) \to 1.$ $\quad -2\log\Lambda_n \xrightarrow{d} NC\,\chi^2(1, (\mu+\varepsilon)^2/\sigma^2).$

③ Behrens – Fisher Problem.

$$\left.\begin{array}{l} X_1 \dots X_m \overset{iid}{\sim} N(\mu_1, \sigma_1^2). \\ Y_1 \dots Y_n \overset{iid}{\sim} N(\mu_2, \sigma_2^2). \end{array}\right] \to H_0 : \mu_1 = \mu_2 \quad vs \quad H_1 : \mu_1 \ne \mu_2.$$

sol).
$$\begin{cases} \hat{\sigma}_1^2 = (\bar{X} - \hat{\mu})^2 + \frac{1}{m}\sum(X_i - \bar{X})^2 \\ \hat{\sigma}_2^2 = (\bar{Y} - \hat{\mu})^2 + \frac{1}{n}\sum(Y_j - \bar{Y})^2. \\ 0 = \frac{m(\bar{X}-\hat{\mu})}{\hat{\sigma}_1^2} + \frac{n(\bar{Y}-\hat{\mu})}{\hat{\sigma}_2^2.} = \left.\frac{\partial l}{\partial \mu}\right|_{\mu=\hat{\mu}.} \end{cases} \qquad \Omega_0 = (\mu, \sigma_1^2, \sigma_2^2).$$

$$\Rightarrow \quad -2\log\Lambda_n = \frac{m}{2}\log\frac{\hat{\sigma}_1^{2\,\Omega}}{\hat{\sigma}_1^{2}{}_{\Omega_0}} + \frac{n}{2}\log\frac{\hat{\sigma}_2^{2\,\Omega}}{\hat{\sigma}_2^{2}{}_{\Omega_0}.}$$

$$= \frac{m}{2}\log\left(\frac{\frac{1}{m}\sum(X_i - \bar{X})^2}{\frac{1}{m}\sum(X_i-\bar{X})^2 + (\bar{X}-\hat{\mu})^2}\right) + \frac{n}{2}\log\left(\frac{\frac{1}{n}\sum(Y_j-\bar{Y})^2}{\frac{1}{n}\sum(Y_j-\bar{Y})^2 + (\bar{Y}-\hat{\mu})^2}\right).$$

↳ σ_1^2/σ_2^2 에 의존. ㊀