<시계열 분석 및 실습>.

Ch1. 시계열 자료

1.1. 추세가 있는 시계열 자료.

① Moving Average Method.

$$X_t = f(t) + \varepsilon_t. \quad \varepsilon_t \sim iid(0, \sigma^2).$$

Let $y_t = \dfrac{X_{t-m+1} + \cdots + X_t}{m}$.

$$y_t \simeq \frac{f(t-m+1) + \cdots + f(t)}{m} \quad \left(\because \frac{\varepsilon_{t-m+1} + \cdots + \varepsilon_t}{m} \xrightarrow[L.L.N.]{} E\varepsilon_1 = 0\right)$$

$$\hat{f}(t) = \frac{X_t + \cdots + X_{t-m+1}}{m} \quad \text{(m이 클수록 smooth).}$$

② Exponential Smoothing Method.

$S_t = \omega X_t + (1-\omega) S_{t-1}$ 로 정의하자.

$$\begin{aligned} S_t &= \omega X_t + (1-\omega)(\omega X_{t-1} + (1-\omega) S_{t-2}) \\ &= \omega X_t + (1-\omega)\omega X_{t-1} + (1-\omega)^2 S_{t-2} \\ &\ \ \vdots \\ &= \omega \sum_{j=0}^{t-1} (1-\omega)^j X_{t-j} + (1-\omega)^t S_0. \\ &\simeq \omega \sum_{j=0}^{t-1} (1-\omega)^j X_{t-j}. \quad (\text{if } S_0 \doteq 0). \end{aligned}$$

$\hat{X}_t = S_{t-1}$ 라 두고 $\underset{\omega}{\operatorname{argmin}} \sum_{t=2}^{n} (X_t - \hat{X}_t)^2 = \omega^*$ 으로 구한다.

1.2. 정상시계열.

Def $\{X_t\}$ is stationary if

(i) $\forall t$, $EX_t = \mu$

(ii) $\forall t$, $EX_t^2 < \infty$, ($Var(X_t)$ exists)

(iii) $\forall t$, $Cov(X_{t+h}, X_t) = Cov(X_0, X_h) = \gamma(h)$ (does not depend on t).

Def $\{X_t\}$ is strictly stationary if $\forall x_1, \ldots, x_n \in \mathbb{R}$, & $h \in \mathbb{Z}$,

$$P(X_1 \le x_1, \ldots, X_n \le x_n) = P(X_{1+h} \le x_1, \ldots, X_{k+h} \le x_k).$$

Note. – 평균, 분산이 존재하지 않으면 강정상 ⇏ 정상 (e.g. Cauchy 분포).
– 강정상 ⊕ $EX_t^2 < \infty$ ⇒ 정상.
– $\{X_t\} \sim IID$ 이면 강정상.

Thm (i) $\gamma(0) \geq 0$
(ii) $\gamma(-h) = \gamma(h)$
(iii) $|\gamma(h)| \leq \gamma(0)$
(iv) $\forall x_1, \dots x_n \in \mathbb{R}$, $t_1, \dots, t_n \in \mathbb{Z}$, $\sum_{i,j=1}^{n} x_i \gamma(t_i - t_j) x_j \geq 0$.

pf) (iii) Use CS : $|EXY| \leq E|XY| \leq \sqrt{EX^2}\sqrt{EY^2}$

$$|Cov(X_t, X_{t+h})| = |E(X_t - \mu)(X_{t+h} - \mu)|$$
$$\leq \sqrt{E(X_t-\mu)^2}\sqrt{E(X_{t+h}-\mu)^2}$$
$$= \sqrt{\gamma(0)} \cdot \sqrt{\gamma(0)}$$

Def $\rho(h) = \dfrac{\gamma(h)}{\gamma(0)}$. : ACF.

Thm (i) $-1 \leq \rho(h) \leq 1$.
(ii) $Corr(X_s, X_t) = Corr(X_t, X_s) = \rho(s-t) = \rho(t-s)$.

Def $\{X_t\} \sim WN(0, \sigma^2)$
if $\gamma(h) = \begin{cases} \sigma^2 & h=0 \\ 0 & h \neq 0. \end{cases}$ → X_t & X_s uncorrelated.

Def $\{X_t\} \sim IID(0, \sigma^2)$
if $\{X_t\} \sim WN(0, \sigma^2)$ & $\{X_t\}$ are iid.

Def $\{X_t\}$ is Gaussian if $\forall n \geq 1$, $(X_1 \dots X_n)$ is a normal random vector

$$\Leftrightarrow f_{X_1 \dots X_n}(x_1 \dots x_n) = \frac{1}{(\sqrt{2\pi})^n |\Sigma|^{1/2}} e^{-\frac{1}{2}(x-\mu)'\Sigma^{-1}(x-\mu)}.$$

$$\left(\begin{array}{l} x = (x_1 \dots x_n)^T \\ \mu = (\mu_1 \dots \mu_n)^T. \\ \Sigma = (\sigma_{ij}) \end{array} \right.$$

Thm If $\{X_t\}$ is Gaussian,

$$E[X_{n+1} \mid \underbrace{X_1 \dots X_n}_{h(X_1 \dots X_n)}) = \sum_{i=1}^{n} a_i X_i.$$

$$\hookrightarrow \underset{a_i}{\operatorname{argmin}} \sum_{t=k+1}^{n} (X_t - a_1 X_{t-1} - \cdots - a_k X_{t-k})^2.$$

Ch2. ARMA 모형.

2.1. 서론 : AR(1), MA(1).

① MA(1).

: $X_t = \varepsilon_t + \theta \varepsilon_{t-1}$. $\theta \in \mathbb{R}$., $\{\varepsilon_t\} \sim WN(0, \sigma^2)$.

(i) $EX_t = E(\varepsilon_t + \theta \varepsilon_{t-1}) = 0$.

(ii) $Var(X_t) = EX_t^2 = E\varepsilon_t^2 + 2\theta E \varepsilon_t \varepsilon_{t-1} + \theta^2 E \varepsilon_{t-1}^2$

$$= (1+\theta^2)\sigma^2$$

(iii) $\gamma(h) = Cov(X_t, X_{t+h})$

$$= Cov(\varepsilon_t + \theta \varepsilon_{t-1}, \varepsilon_{t+h} + \theta \varepsilon_{t+h-1})$$

$$= \begin{cases} \theta\sigma^2 & \text{if } h=1 \\ 0 & \text{if } h \ge 2. \end{cases}$$

$\Rightarrow X_t$ is stationary.

(iv) $\rho(h) = \gamma(h)/\gamma(0) = \begin{cases} 1 & h=0 \\ \frac{\theta}{1+\theta^2} & h=1 \\ 0 & h \ge 2. \end{cases}$

(v) If $\{\varepsilon_t\} \sim IID$, X_t is strictly stationary.

$(X_1 \dots X_n) \overset{d}{=} (X_{1+h}, \dots X_{n+h})$.

② AR(1)

: $X_t = \phi X_{t-1} + \varepsilon_t$, $\phi \in \mathbb{R}$

(a) If $\phi = 1$,

$$\begin{cases} X_1 = X_0 + \varepsilon_1 \\ X_2 = X_1 + \varepsilon_2 = X_0 + \varepsilon_1 + \varepsilon_2 \\ \vdots \\ X_t = X_0 + \varepsilon_1 + \cdots + \varepsilon_t. \end{cases}$$

(i) X_t is not stationary.

pf) Suppose X_t is stationary.

$$Var(X_t - X_0) = Var(\varepsilon_1 + \cdots + \varepsilon_t).$$

$$\begin{aligned}(LHS) &= Var(X_t) - 2Cov(X_t, X_0) + Var(X_0) \\ &= 2(\gamma(0) - \gamma(t)). \\ &\le 4\gamma(0) \quad (\because |\gamma(t)| \le \gamma(0)) \\ &< \infty. \quad \forall t.\end{aligned}$$

$(RHS) = t\sigma^2 \to \infty$ as $t \to \infty$. $\Rightarrow\!\Leftarrow$. ■

(b) $\phi = -1$.

$$X_t = X_0 - \varepsilon_1 + \cdots + (-1)^t \varepsilon_t.$$

$\Leftrightarrow X_t - X_0 = -\varepsilon_1 + \cdots + (-1)^t \varepsilon_t \rightarrow$ non-stationary.

(c) $|\phi| < 1$.

$$\begin{aligned}X_t &= \phi X_{t-1} + \varepsilon_t. \\ &= \phi(\phi X_{t-2} + \varepsilon_{t-1}) + \varepsilon_t \\ &= \phi^2 X_{t-2} + \phi\varepsilon_{t-1} + \varepsilon_t \\ &\vdots \\ &= \underbrace{\phi^n X_{t-n}}_{①} + \underbrace{\sum_{j=0}^{n-1} \phi^j \varepsilon_{t-j}}_{②}.\end{aligned}$$

If $n \to \infty$,

$$\begin{aligned}① : E|\phi^n X_{t-n}| &= |\phi^n| E|X_{t-n}| \\ &\le |\phi|^n (\underbrace{E X_{t-n}^2}_{=\gamma(0)})^{\frac{1}{2}} \to 0.\end{aligned}$$

$$\begin{aligned}② \to E\left|\sum_{j=0}^{\infty} \phi^j \varepsilon_{t-j}\right| &\le E\left(\sum_{j=0}^{\infty} |\phi|^j |\varepsilon_{t-j}|\right) \\ &= \sum_{j=0}^{\infty} E(|\phi|^j |\varepsilon_{t-j}|) \\ &= \sum_{j=0}^{\infty} |\phi|^j E|\varepsilon_{t-j}| \le E(\varepsilon_{t-j}^2)^{\frac{1}{2}} \\ &< \infty.\end{aligned}$$

$\therefore \sum_{j=0}^{\infty} \phi^j \varepsilon_{t-j}$ exists w.p. 1.

(* If $E|X| < \infty$, $P(|X| < \infty) = 1$ ($|X| < \infty$ w.p.1.)

$\because P(|X| \ge c) \le \frac{E|X|}{c} \Rightarrow \underbrace{\lim P(|X| \ge n)}_{0} = P(\cap(|X| \ge n)) = P(|X| = \infty)$).

$X_t = \sum_{j=0}^{\infty} \phi^j \varepsilon_{t-j}$. $\longrightarrow$ $X_t = \phi X_{t-1} + \varepsilon_t$

$(X_{t+1} = \phi X_t + \varepsilon_{t+1})$.

pf) $X_{t+1} = \sum_{j=0}^{\infty} \phi^j \varepsilon_{t+1-j}$

$= \varepsilon_{t+1} + \phi \sum_{j=1}^{\infty} \phi^{j-1} \varepsilon_{t+1-j}$

Letting $j-1 = i$,

$= \varepsilon_{t+1} + \phi \sum_{i=1}^{\infty} \phi^i \varepsilon_{t-i}$. ☒

이제 stationarity check

(i) $EX_t = \sum_{j=1}^{\infty} \phi^j \varepsilon_{t-j} = 0$.

(ii) $var(X_t) = EX_t^2$

$= Cov(\sum \phi^j \varepsilon_{t-j}, \sum \phi^i \varepsilon_{t-i})$

$= \sum_{i,j} \phi^i \phi^j Cov(\varepsilon_{t-j}, \varepsilon_{t-i})$ ↳ $j=i$인 term만.

$= \sum_{i=0}^{\infty} \phi^{2i} \sigma^2$

$= \frac{\sigma^2}{1-\phi^2} < \infty$.

(iii) $Cov(X_t, X_{t+h}) = Cov(\sum \phi^i \varepsilon_{t-i}, \sum \phi^j \varepsilon_{t+h-j})$

$= \sum_{i,j} \phi^i \phi^j Cov(\varepsilon_{t-i}, \varepsilon_{t+h-j})$ ↳ $i = j-h$.

$= \sum_i \phi^i \phi^{i+h} \sigma^2$

$= \frac{\phi^h \sigma^2}{1-\phi^2}$.

$\therefore \gamma(h) = \frac{\phi^h \sigma^2}{1-\phi^2}$ $h = 0, 1, 2, \ldots$

+) $\rho(h) = \phi^h \longrightarrow 0$ as $h \to \infty$.

$\Rightarrow$ $X_t = \sum \phi^j \varepsilon_{t-j}$ is a stationary proc.

(unique sol.).

(d) $|\phi| > 1$.

$X_t = \phi X_{t-1} + \varepsilon_t$ $\Rightarrow$ explosive proc.

Let $X_t = \sum_{j=1}^{\infty} \phi^{-j} \varepsilon_{t+j}$ $\rightarrow$ strictly stationary sol.

$(|\phi| > 1 \Leftrightarrow |\phi^{-1}| < 1)$.

Def $\{X_t\}$ is a linear process

when $X_t = \sum_{i=0}^{\infty} a_i \varepsilon_{t-i}$, $\{\varepsilon_t\} \sim iid(0, \sigma^2)$.

where a_i are real numbers with $\sum_{i=0}^{\infty} |a_i| < \infty$.

thm (i) $\sum_{i=0}^{\infty} a_i \varepsilon_{t-i}$ converges w.p. 1.

pf) $E|\sum a_i \varepsilon_{t-i}| \le E\sum |a_i||\varepsilon_{t-i}|$

$= \sum |a_i| E|\varepsilon_{t-i}| \le \sqrt{E\varepsilon_{t-i}^2} < \infty.$

$\Leftrightarrow P(\sum_{i=0}^{\infty} |a_i||\varepsilon_{t-i}| < \infty) = 1.$ ∎

(ii) $EX_t = E(\sum a_i \varepsilon_{t-i}) = \sum a_i E\varepsilon_{t-i} = 0.$

(iii) $EX_t^2 = E(\sum a_i \varepsilon_{t-i})^2$

$= E \sum a_i^2 \varepsilon_{t-i}^2 + 2E\sum_{i<j} a_i a_j \varepsilon_{t-i}\varepsilon_{t-j}.$

$= \sum a_i^2 \sigma^2 < \infty.$

(iv) $\gamma(h) = Cov(\sum a_i \varepsilon_{t-i}, \sum a_j \varepsilon_{t+h-j}$

$= \sum\sum a_i a_j Cov(\varepsilon_{t-i}, \varepsilon_{t+h-j})$ $i = j-h$

$= \sum a_i a_{i+h} \sigma^2$

$\Rightarrow$ stationary.

(& strictly stationary).

thm If $\{X_t\}$ is stationary & $\{b_i\}$ is a real seq s.t. $\sum_{i=0}^{\infty}|b_i| < \infty$, $Y_t := \sum_{i=0}^{\infty} b_i X_{t-i}$ is also a stationary process.

pf) (i) $EY_t = E(\sum b_i X_{t-i}) = \sum b_i \underbrace{EX_{t-i}}_{=\mu}$ $\forall t$. (t에 무관).

(ii) $EY_t^2 = Cov(Y_t, Y_t)$.

(iii) $\gamma_Y(h) = Cov(Y_t, Y_{t+h}) = Cov(\sum b_i X_{t-i}, \sum b_j X_{t+h-j})$.

$= \sum b_i b_j Cov(X_{t-i}, X_{t+h-j})$

$= \sum b_i b_j \gamma_X(h-j+i)$. $\forall t$ (t에 무관).

∎

Def B is the back shift operator: $Bf(t) = f(t-1)$.

Thm
- $BX_t = X_{t-1}$.
- $Bc = c$.
- $B(X_t + Y_t) = X_{t-1} + Y_{t-1}$.
- $B(\alpha X_t + \beta Y_t) = \alpha X_{t-1} + \beta Y_{t-1}$
  $= \alpha BX_t + \beta BY_t$ (linearity).
- $B^2 f(t) = B \cdot f(t-1) = f(t-2)$.

Def $f(t) = a_0 + a_1 t + \cdots + a_p t^p$
$\Rightarrow f(B) = a_0 + a_1 B + \cdots + a_r B^p$.

Thm . $f(B)X_t = a_0 X_t + a_1 X_{t-1} + \cdots + a_p X_{t-p}$.

Thm Assume $\sum_{i=0}^{\infty} |a_i| < \infty$. $\sum_{i=0}^{\infty} |b_i| < \infty$.
If $\{X_t\}$ is stationary,

$$X_t = a(B)(b(B)X_t) = (a \cdot b)(B) X_t \quad \text{is also stationary.}$$
$$= b(B)(a(B)X_t)$$

where $a(z) = \sum_{i=0}^{\infty} a_i z^i$, $b(z) = \sum_{j=0}^{\infty} b_j z^j$.

pf) Note that $a(B)(b(B)X_t) = (\sum a_i B^i)(\sum b_j \underbrace{B^j X_t}_{X_{t-j}})$

$$= \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} a_i b_j B^i X_{t-j}$$
$$= \sum\sum a_i b_j X_{t-(i+j)}. \quad i+j=k. \; j=k-i \geq 0.$$
$$= \sum_{k=0}^{\infty}\left(\sum_{i=0}^{k} a_i b_{k-i}\right) X_{t-k}.$$

($a(z) \cdot b(z) \to z^k$의 계수를 $(a\cdot b)_k$ 라 표현하자.
$(a \cdot b)_k = \sum_{i=0}^{k} a_i b_{k-i}$)

$$= \sum_{k=0}^{\infty} (a\cdot b)_k X_{t-k}.$$

(이때, $\sum_{k=0}^{\infty} |(a\cdot b)_k| < \infty$ 이다.)

$$= (a \cdot b)(B) X_t.$$

$\to$ stationary

Ex. ① AR(1)

$$X_t = \phi X_{t-1} + \varepsilon_t$$

$$\Leftrightarrow X_t - \phi X_{t-1} = \varepsilon_t.$$

$$\Leftrightarrow (1-\phi B) X_t = \varepsilon_t \Rightarrow \phi(B) X_t = \varepsilon_t.$$

(define $\phi(z) = 1 - \phi z$. $|z| \le 1$, $z \in \mathbb{C}$).

observe that

$$\frac{1}{\phi(z)} = \frac{1}{1-\phi z} = \sum_{i=0}^{\infty} \phi^i z^i \Rightarrow \frac{1}{\phi(B)} = \sum_{i=0}^{\infty} \phi^i B^i.$$

$$\frac{1}{\phi(B)} \phi(B) X_t = \frac{1}{\phi(B)} \varepsilon_t.$$

$$\Leftrightarrow \left(\frac{1}{\phi}\cdot\phi\right)(B) X_t = \sum \phi^i B^i \varepsilon_t$$

$$\Leftrightarrow X_t = \sum \phi^i \varepsilon_{t-i}.$$ → AR(1)의 sol.

② MA(1)

$$X_t = \varepsilon_t + \theta \varepsilon_{t-1}.$$

$$\Leftrightarrow X_t = \theta(B) \varepsilon_t.$$

(Define $\theta(z) = 1 + \theta z$, $|z| \le 1$).

$$\frac{1}{\theta(z)} = \frac{1}{1+\theta(z)} = \sum_{i=0}^{\infty} (-\theta)^i z^i$$

$$\frac{1}{\theta(B)} X_t = \frac{1}{\theta(B)} \theta(B) \varepsilon_t$$

$$\Leftrightarrow \frac{1}{\theta(B)} X_t = \varepsilon_t$$

$$\Leftrightarrow \sum_{i=0}^{\infty} (-\theta)^i B^i X_t = \varepsilon_t.$$

$$\Leftrightarrow X_t = \theta X_{t-1} - \theta^2 X_{t-2} + \theta^3 X_{t-3} + \cdots + \varepsilon_t.$$

$$\simeq \theta X_{t-1} + \cdots + \theta^p (-1)^{p+1} X_{t-p}.$$

→ $t-1 \ldots t-p$ 를 통해 X_t forecasting 가능.

(Def X_t는 $|\theta| < 1$ 일 때 Invertible).

ㄴ ε_t를 X_t로 표현가능.

∵ $\theta(z) \ne 0$ $\forall |z| \le 1$.

$$\Rightarrow 1 + \theta z = 0 \Leftrightarrow z = -\frac{1}{\theta} \Leftrightarrow \left|-\frac{1}{\theta}\right| > 1 \Leftrightarrow |\theta| < 1.$$

Def $\{X_t\}$ is called an $AR(p)$ process

if $X_t = \phi_1 X_{t-1} + \cdots + \phi_p X_{t-p} + \varepsilon_t$. $\{\varepsilon_t\} \sim iid(0, \sigma^2)$.

- $\phi(z) = 1 - \phi_1 z - \cdots - \phi_p z^p$ 라 두면, $(|z| \le 1)$

$\phi(B) X_t = \varepsilon_t$.

$\Leftrightarrow \frac{1}{\phi(B)} \phi(B) X_t = \frac{1}{\phi(B)} \varepsilon_t$.

($\phi(z) \neq 0$, $\forall |z| \le 1$ 이면 $\frac{1}{\phi(z)} = \sum a_i z^i$ $\forall |z| \le 1$ 로 표현가능. 이때 a_i는 $\sum |a_i| < \infty$.)

$\Leftrightarrow X_t = \sum_{i=0}^{\infty} a_i \varepsilon_{t-i}$

$a_i = a_i(\underbrace{\phi_1 \ldots \phi_p}_{a_0 = 1})$, $\left(\because \phi(z) \cdot \sum_{i=1}^{\infty} a_i z^i \equiv 1\right)$.

Def $\{X_t\}$ is an $MA(q)$ process

if $X_t = \varepsilon_t + \theta_1 \varepsilon_{t-1} + \cdots + \theta_q \varepsilon_{t-q}$.

- $\theta(z) = 1 + \theta_1 z + \cdots + \theta_q z^q$ 라 두면

$X_t = \theta(B) \varepsilon_t$.

$\Leftrightarrow \frac{1}{\theta(B)} X_t = \varepsilon_t$.

($\theta(z) \neq 0$, $\forall |z| \le 1$ 이면 $\frac{1}{\theta(z)} = \sum \pi_i z^i$ 로 표현가능,

$\Leftrightarrow \sum_{i=0}^{\infty} \pi_i X_{t-i} = \varepsilon_t$.

$\pi_i = \pi_i(\theta_1, \ldots \theta_t)$. $\rightarrow$ $\sum a_i b_{k-i}$ 로 표현가능.

invertibility!

Def $\{X_t\}$ is defined as $ARMA(p,q)$ process

if $X_t - \phi_1 X_{t-1} - \cdots - \phi_p X_{t-p} = \varepsilon_t + \theta_1 \varepsilon_{t-1} + \cdots + \theta_q \varepsilon_{t-q}$.

- Define $\begin{cases} \phi(z) = 1 - \phi_1 z^1 - \cdots - \phi_p z^p \\ \theta(z) = 1 + \theta_1 z^1 + \cdots + \theta_q z^q. \end{cases}$

$\Rightarrow \phi(B) X_t = \theta(B) \varepsilon_t$. 로 표현가능.

$\rightarrow \frac{1}{\phi(z)}$ - 유리함수.

<u>Thm</u> $\{X_t\}$ is stationary & causal if $\phi(z) \neq 0$ $(\forall |z| \le 1)$.

pf) $\frac{1}{\phi(B)}\phi(B) X_t = \frac{1}{\phi(B)}\theta(B)\varepsilon_t$

$\Leftrightarrow \quad X_t = \left(\frac{\theta}{\phi}\right)(B)\varepsilon_t = \underbrace{\left(\sum_{i=1}^{\infty} a_i B^i\right)}_{\text{로 표현}}\varepsilon_t$

$= \sum_{i=1}^{\infty} a_i \varepsilon_{t-i}.$ $\quad (a_i = f(\phi_1 \dots \phi_p; \theta_1 \dots \theta_q))$.

$\rightarrow$ Linear proc. 이고 $\{\varepsilon_t\} \sim \text{iid}(0, \sigma^2)$ 이므로

$\{X_t\}$는 strictly stationary.

$\left(\gamma_X(h) = \left(\sum_{i=0}^{\infty} a_i a_{i+h}\right)\sigma^2.\right)$

<u>Thm</u> $\{X_t\}$ is invertible if $\theta(z) \neq 0$ $(\forall |z| \le 1)$

pf) $\underbrace{\left(\frac{\phi}{\theta}\right)}(B) X_t = \varepsilon_t$.

$\hookrightarrow \pi(B) = \sum \pi_i B^i, \quad \pi_0 = 1.$

$\Leftrightarrow \quad X_t + \pi_1 X_{t-1} + \dots + \dots = \varepsilon_t.$ $\quad$ (forecasting 가능한 이론적 근거).

<u>Def</u> (Partial ACF.) : k의 함수.

Let $\{X_t\}$ be a stationary process with mean $= 0$.

& $F = \text{span}\{X_2 \dots X_k\} = \{\sum_{i=2}^{k} a_i X_i, \ a_i \in \mathbb{R}\}$.

$\phi_{kk} = \text{Corr}(X_{k+1} - P_F X_{k+1}, X_1 - P_F X_1)$. $k \ge 1$.

* <u>F의 성질</u> (i) If $y_n \in F$ & $E|y_n - y^*|^2 \xrightarrow{(n\to\infty)} 0$, $(y^* = \lim y_n)$.

$\rightarrow y^* \in F$. (denote as $\sum a_i^* X_i$).

(ii) If $y, z \in F$, $ay + bz \in F$.

(* <u>Projection theory</u> $EY^2 < \infty$, $Y = Y_1 + Y_2$ $\in F$, $Y_2 \perp F$.

$EY_2 \cdot Z = 0, \quad \forall z \in F$)

(iii) for any Y with $EY^2 < \infty$,

$\exists$ unique $Z = P_F Y$ s.t. $E(Y - Z) \cdot X_l = 0$. $(l = 2, \dots, k)$.

<u>구하는 방법</u>. $\begin{pmatrix} \rho(0) & \rho(1) & \cdots & \rho(h-1) \\ \rho(1) & \rho(0) & \cdots & \rho(h-2) \\ \vdots & & \ddots & \vdots \\ \rho(h-1) & & \cdots & \rho(0) \end{pmatrix} \begin{pmatrix} \phi_{k1} \\ \vdots \\ \underbrace{\phi_{kk}}_{\text{부분 자기상관계수}} \end{pmatrix} = \begin{pmatrix} \rho(1) \\ \vdots \\ \rho(h) \end{pmatrix}$. $\Leftrightarrow \Gamma\phi = \rho$.

$\rightarrow \phi = \Gamma^{-1}\rho$.

AR(p) 모형은 $h > p$ 에서 $\phi_{hh} = 0$.
MA(q) 모형은 $h > q$ 에서 $\gamma(h) = 0$. $\Big\} \to$ 주어진 sample에서 p, q 추정 가능.

Ch3. 정상 ARMA 모형에서의 추론.

3.1. 평균의 추정.

- Assume $\{X_t\} \sim ARMA(p,q)$. with mean $= \mu$,

$$\phi(B)(X-\mu) = \theta(B)\varepsilon_t.$$

$$\Leftrightarrow X-\mu = \left(\frac{\theta}{\phi}\right)(B)\varepsilon_t$$
$$= \sum a_i B^i \varepsilon_t$$
$$= \sum a_i \varepsilon_{t-i}.$$

$\begin{cases} \text{(i) } \sum |a_i| < \infty \\ \text{(ii) } b_j = \sum_{k \ge j+1}^{\infty} a_k < \infty. \end{cases}$ ※

- We can write

$$X_t = \mu + a_0\varepsilon_t + \sum_{i=1}^{\infty}(b_{i-1} - b_i)\varepsilon_{t-i}$$
$$= \mu + \left(\sum_{i=0}^{\infty} a_i\right)\varepsilon_t - \underbrace{\left(\sum_{i=1}^{\infty} a_i\right)}_{b_0}\varepsilon_t + \sum_{i=1}^{\infty} b_{i-1}\varepsilon_{t-i} - \sum_{i=1}^{\infty} b_i\varepsilon_{t-i}$$
$$= \mu + \left(\sum_{i=0}^{\infty} a_i\right)\varepsilon_t + \sum_{j=0}^{\infty} b_j\varepsilon_{t-j-1} - \sum_{i=0}^{\infty} b_i\varepsilon_{t-i}. \quad (i-1=j)$$

$\left(y_t := \sum_{i=0}^{\infty} b_i\varepsilon_{t-i} \text{ 로 정의하면}\right)$ $\to$ y_t는 stationary.

$$= \mu + \left(\sum_{i=0}^{\infty} a_i\right)\varepsilon_t + y_{t-1} - y_t.$$

$$\Leftrightarrow X_t - \mu = \left(\sum_{i=0}^{\infty} a_i\right)\varepsilon_t + y_{t-1} - y_t.$$

$\hat{\mu} = \frac{1}{n}\sum_{t=1}^{n} X_t$ 로 두면,

$$\sqrt{n}(\hat{\mu} - \mu) = \frac{1}{\sqrt{n}}\sum_{t=1}^{n}(X_t - \mu)$$
$$= \underbrace{\left(\sum_{i=1}^{\infty} a_i\right)\frac{1}{\sqrt{n}}\sum_{t=1}^{n}\varepsilon_t}_{\xrightarrow{d} N\left(0, \left(\sum_{i=1}^{\infty} a_i\right)^2\sigma^2\right)} + \underbrace{\frac{1}{\sqrt{n}}\sum_{t=1}^{n}(y_{t-1} - y_t)}_{\to 0}.$$

$\left(\because \frac{Ey_n}{\sqrt{n}} = \frac{Ey_0}{\sqrt{n}} = 0\right)$

Ex AR(1) : $X_t = \phi X_{t-1} + \varepsilon_t$. $\rightarrow \alpha_i = \phi^i$ 이므로

$$\sqrt{n}(\hat{\mu} - \mu) \overset{d}{\rightarrow} N\left(0, \frac{\sigma^2}{(1-\phi)^2}\right).$$

(conf. int : $\hat{\mu} \pm z_{\alpha/2} |\Sigma \alpha_i| \sigma$).

3.2. Sample ACF (SACF).

Def
$$\hat{\gamma}(h) = \begin{cases} \frac{1}{n} \sum_{t=1}^{n-h} (X_t - \bar{X})(X_{t+h} - \bar{X}). & h \geq 0, \ h \leq n-1. \\ \hat{\gamma}(-h) & h < 0. \end{cases}$$

($P(\hat{\gamma}(h) \rightarrow \gamma(h), n \rightarrow \infty) = 1$ 임이 알려져 있다.)

Def
$$\hat{\rho}(h) = \frac{\hat{\gamma}(h)}{\hat{\gamma}(0)}. \quad (\longrightarrow \rho(h))$$

Def
$$\begin{pmatrix} \hat{\rho}(0) & \hat{\rho}(1) & \cdots & \hat{\rho}(h-1) \\ \hat{\rho}(1) & \hat{\rho}(0) & \cdots & \hat{\rho}(h-2) \\ \vdots & & & \vdots \\ \hat{\rho}(h-1) & & & \hat{\rho}(0) \end{pmatrix} \begin{pmatrix} \hat{\phi}_{h1} \\ \vdots \\ \hat{\phi}_{hh} \end{pmatrix} = \begin{pmatrix} \hat{\rho}(1) \\ \vdots \\ \hat{\rho}(h) \end{pmatrix}$$

$\hat{\phi}_{hh} \hookrightarrow$ SPACF. $(\longrightarrow \phi_{hh})$.

Thm (Bartlett's Formula).

If $X_t = \mu + \sum_{i=0}^{\infty} \alpha_i \varepsilon_{t-i}$, $E\varepsilon_i^4 < \infty$,

$\sqrt{n}(\hat{\rho}(h) - \rho(h)) \overset{d}{\rightarrow} N(0, W)$.

$\left(\begin{array}{l} \hat{\rho}(h) = (\hat{\rho}(1) \cdots \hat{\rho}(h))^T \\ \rho(h) = (\rho(1) \cdots \rho(h))^T. \end{array} \right.$

where $W = (w_{ij})$

$$w_{ij} = \sum_{k=0}^{\infty} (\rho(k+i) \cdots\cdots).$$

3.3. 자료추정법. (in AR(p))

$$X_t = \phi_1 X_{t-1} + \cdots + \phi_p X_{t-p} + \varepsilon_t.$$

여기서 $E(X_t - \phi_1 X_{t-1} - \cdots - \phi_p X_{t-p}) X_{t-j} = E\varepsilon_t X_{t-j} = 0$ 이므로

$\gamma(j) = \phi_1 \gamma(j-1) + \cdots + \phi_p \gamma(j-p)$ 가 성립한다.

$\rightarrow$ (행렬로)
$$\begin{pmatrix} \rho(0) & \rho(1) & \cdots & \rho(p-1) \\ \rho(1) & \rho(0) & & \rho(p-2) \\ \vdots & & & \vdots \\ \rho(p-1) & \rho(p-2) & & \rho(0) \end{pmatrix} \begin{pmatrix} \phi_1 \\ \vdots \\ \phi_p \end{pmatrix} = \begin{pmatrix} \rho(1) \\ \vdots \\ \rho(p) \end{pmatrix}$$

$\Gamma \underset{\sim}{\phi} = \gamma$ 의 근 $\underset{\sim}{\phi} \to \hat{\phi} = \hat{\Gamma}^{-1}\hat{\gamma}$: $\overset{Y-W}{\text{적률추정량}}$

$(P(\hat{\phi}_{YW} \to \phi,\ n \to \infty) = 1)$

3.4. Least Squares Estimator.

① AR(1). $X_t = \phi X_{t-1} + \varepsilon_t$. $(X_0 = 0)$

$$\Rightarrow \sum_{t=1}^{n} (X_t - \phi X_{t-1})^2 \text{ 를 최소로 하는}$$

$$\hat{\phi} = \frac{\sum_{t=1}^{n} X_{t-1} X_t}{\sum_{t=1}^{n} X_{t-1}^2}$$

$$\sqrt{n}(\hat{\phi} - \phi) = \sqrt{n} \cdot \frac{\sum X_{t-1} X_t - \phi \sum_{t=1}^{n} X_{t-1}^2}{\sum X_{t-1}^2} \to \sum X_{t-1} \underbrace{(X_t - \phi X_{t-1})}_{\varepsilon_t}$$

$$= \frac{\frac{1}{\sqrt{n}} \sum_{t=1}^{n} X_{t-1} \varepsilon_t \to N(0, \gamma(0)\sigma^2). \text{ (known)}}{\underbrace{\frac{1}{n} \sum_{t=1}^{n} X_{t-1}^2}\ \to \gamma(0)}$$

$$\xrightarrow{d} N(0, \gamma(0)^{-1}\sigma^2).$$

①' AR(p): $X_t = \phi_1 X_{t-1} + \cdots + \phi_p X_{t-p} + \varepsilon_t$.

Let $\underset{\sim}{X_t} = (X_t, \ldots, X_{t-p+1})^T$

$$\Rightarrow \hat{\phi} = \left(\sum_{t=1}^{n} \underset{\sim}{X_{t-1}} \underset{\sim}{X_{t-1}}^T \right)^{-1} \sum_{t=1}^{n} \underset{\sim}{X_{t-1}} \underset{\sim}{X_t} \quad (\underset{\text{이라고 가정}}{x_0 = \cdots = x_p = 0}).$$

$$\sqrt{n}(\hat{\phi} - \phi) = \left(\frac{\sum X_{t-1} X_{t-1}^T}{\sqrt{n}} \right)^{-1} \cdot \frac{\sum X_{t-1} \varepsilon_t}{n} \xrightarrow{d} \Gamma^{-1} N(0, \Gamma\sigma^2).$$
$$= N(0, \Gamma^{-1}\sigma^2).$$

$(E\underset{\sim}{X_i}\underset{\sim}{X_j}^T = \gamma(i-j) = \Gamma,\ Cov(\underset{\sim}{X_{t-1}}\varepsilon_t, \underset{\sim}{X_{t-1}}\varepsilon_t) = \Gamma\sigma^2).$

3.5. Conditional MLE.

① AR(1): $X_t = \phi X_{t-1} + \varepsilon_t$. $\varepsilon_t \sim$ iid $N(0, \sigma^2)$

↳ If X_{t-1} given, $X_t \sim N(\phi X_{t-1}, \sigma^2)$.

$$f(X_0, \ldots X_n) = f(X_n | X_{n-1} \cancel{\ldots X_0}) \times \cdots \times f(X_1 | X_0) f(X_0).$$

$$= \prod_{t=2}^{n} f(X_t | X_{t-1}) \cdot f(X_0).$$

↳ conditioning으로 제거.

Since $f(X_t | X_{t-1}) = \frac{1}{\sqrt{2\pi\sigma}} \exp\left(-\frac{(X_t - \phi X_{t-1})^2}{2\sigma^2}\right)$,

$$L(\phi, \sigma^2) = -\frac{n-1}{2}\log(2\pi\sigma) - \frac{1}{2\sigma^2}\sum_{t=2}^{n}(X_t - \phi X_{t-1})^2.$$

$$\Rightarrow \quad \hat{\phi} = \underset{(= LSE).}{\frac{\Sigma X_{t-1} X_t}{\Sigma X_{t-1}^2}}, \quad \sigma^2 = \frac{1}{n-1}\Sigma(X_t - \hat{\phi} X_{t-1})^2$$

3.6. Prediction.

① $AR(1)$: $X_t = \phi X_{t-1} + \varepsilon_t. \rightarrow X_t = \sum_{j=0}^{\infty} \phi^j \varepsilon_{t-j}.$

(i)
$$\begin{aligned}
\hat{X}_t(1) &= E(X_{t+1} | X_t \ldots X_1) \\
&= E(\phi X_t + \varepsilon_{t+1} | X_t \ldots X_1) \\
&= \underbrace{\phi E(X_t | X_t \ldots X_1)}_{= \phi E(X_t).} + \underbrace{E(\varepsilon_{t+1} | X_t \ldots X_1)}_{= E(\varepsilon_{t+1})} \\
&= \phi X_t.
\end{aligned}$$

(ii)
$$\begin{aligned}
\hat{X}_t(2) &= E(X_{t+2} | X_t, \ldots, X_1) \\
&= E(\phi X_{t+1} + \varepsilon_{t+2} | X_t \ldots X_1) \\
&= \phi E(X_{t+1} | X_t \ldots X_1) \\
&= \phi \hat{X}_t(1) \\
&= \phi^2 X_t.
\end{aligned}$$

(iii)
$$\begin{aligned}
\hat{X}_t(h) &= E(X_{t+h} | X_t \ldots X_1) \\
&= \phi \hat{X}_t(h-1) = \phi^2 \hat{X}_t(h-2) \\
&= \cdots = \phi^h X_t.
\end{aligned}$$

↳ Let $\tilde{X}_t = X_t - \mu$.

② $AR(1)$ with mean μ : $X_t - \mu = \phi(X_{t-1} - \mu) + \varepsilon_t$.

(i) $\hat{\tilde{X}}_t(1) = E(\tilde{X}_{t+1} | \tilde{X}_t, \ldots, \tilde{X}_1) = \phi(X_t - \mu)$.

$\Leftrightarrow \hat{X}_t(1) = E(X_{t+1} | \tilde{X}_t \ldots \tilde{X}_1) = \mu + \phi(X_t - \mu)$.

$\vdots$

(ii) $\hat{X}_t(h) = \mu + \phi^h (X_t - \mu)$.

③ $ARMA(1,1)$: $X_t = \phi X_{t-1} + \varepsilon_t + \theta \varepsilon_{t-1}$. $|\phi| < 1$, $|\theta| < 1$.

(i) $\hat{X}_t(1) = E(X_{t+1} | X_t \ldots X_1)$

$\triangleq E(X_{t+1} \mid X_t \dots)$ for large t.

$= E(\phi X_t + \varepsilon_{t+1} + \theta \varepsilon_t \mid \text{''})$

$= \phi E(X_t \mid X_t \dots) \rightarrow \phi X_t$

$+ E(\varepsilon_{t+1} \mid \dots) \rightarrow 0$

$+ \theta E(\varepsilon_t \mid \dots) \rightarrow \theta \varepsilon_t$. ($\because \varepsilon_t$는 $X_t \dots X_{t-1}$의 함수).

$\simeq \phi X_t + \theta \varepsilon_t$.

$\hat{\varepsilon}_t = X_t - \hat{\phi} X_{t-1} - \hat{\theta}\hat{\varepsilon}_{t-1} \dots$ ($\hat{\varepsilon}_0$으로 두고 순차적으로 구한다.)

(ii) $\hat{X}_t(2) \triangleq E(X_{t+2} \mid X_t \dots)$

$= E(\phi X_{t+1} + \underbrace{\varepsilon_{t+2} + \theta\varepsilon_{t+1} \mid \text{''}}_{\text{indep.}})$

$= \phi E(X_{t+1} \mid \dots)$

$= \phi \hat{X}_t(1)$

(iii) $\hat{X}_t(h) = \phi^{h-1} \hat{X}_t(1)$

$= \phi^h X_t + \phi^{h-1} \varepsilon_t$

* Best Linear Predictor? $E(X_{n+h} \mid X_1 \dots X_n) \rightarrow f(X_1 \dots X_n)$

$E\left(X_{n+h} - \sum_{i=1}^{n} a_i X_i\right)^2$을 최소로 하는 a_i

$\Leftrightarrow \gamma(n+h-j) = \sum a_i \gamma(j-i)$의 해를 구한다.

Ch4. 비정상시계열 및 모형선택이론

4.1. ARIMA 모형.

<u>Def</u> $\{X_t\} \sim ARIMA(p,d,q)$ if $(1-B)^d X_t$ is stationary and follows an $ARMA(p,q)$ model.

$\Rightarrow \phi(B)(1-B)^d X_t = \theta(B)\varepsilon_t$. $\begin{cases} \phi(z) = 1 - \phi_1 z^1 - \dots - \phi_p z^p \\ \theta(z) = 1 + \theta_1 z^1 + \dots + \theta_q z^q \end{cases}$

<u>Ex</u> $X_t = \varepsilon_t + \underset{\sim}{t}$ deterministic trend

$(1-B)X_t = X_t - X_{t-1}$

$= \varepsilon_t + t - (\varepsilon_{t-1} + (t-1))$

$= \varepsilon_t - \varepsilon_{t-1} + 1$

$\Leftrightarrow (1-B)X_{t-1} = \varepsilon_t - \varepsilon_{t-1}$: $ARMA(0,1,1)$.

4.2. Unit Root Test.

$X_t = \phi X_{t-1} + \varepsilon_t$. $\quad (X_0 \equiv 0)$

↳ If $\phi = 1$, $\quad X_t = X_{t-1} + \varepsilon_t$ (random walk).

(i) $H_0 : \phi = 1$

$H_1 : |\phi| < 1$, $\quad \hat{\phi}_{LS} = \dfrac{\sum X_{t-1} X_t}{\sum X_{t-1}^2} \overset{?}{\approx} 1$

(ii) Test Statistic : $T_n = n(\hat{\phi} - 1)$.

↳ If $|T_n|$ is large, we reject H_0 $\left(\begin{array}{l} |T_n| > c. \\ c : P(|T_n| > c \mid H_0) = \alpha \\ \simeq P(|T| > c_\alpha). \end{array} \right)$

(a) Under H_0,

$$T_n = n\left(\frac{\sum X_{t-1} X_t}{\sum X_{t-1}^2} - 1 \right)$$

$$= n\left(\frac{\sum X_{t-1}(X_t - X_{t-1})}{\sum X_{t-1}^2} \right)$$

$$= \cancel{n}\left(\frac{\sum X_{t-1}\varepsilon_t}{\sum X_{t-1}^2} \right) \cdot \frac{\frac{1}{n\sigma^2}}{\frac{1}{n^2\sigma^2}}$$

$$= \frac{\sum_{t=1}^{n} \frac{X_{t-1}}{\sqrt{n}\sigma} \frac{\varepsilon_t}{\sqrt{n}\sigma}}{\frac{1}{n}\sum_{t=1}^{n} \left(\frac{X_{t-1}}{\sqrt{n}\sigma} \right)^2}$$

* $W_n(s) = \dfrac{1}{\sqrt{n}\sigma} \sum_{t=1}^{[ns]} \varepsilon_t, \quad 0 \le s \le 1$

(i) $E W_n(s) = 0$, $\quad W_n(0) = 0$.

(ii) $\forall\, 0 \le s < t \le 1$,

$$W_n(t) - W_n(s) = \frac{1}{\sqrt{n}\sigma} \sum_{i=[ns]+1}^{[nt]} \varepsilon_i$$

$$\overset{d}{=} \frac{1}{\sqrt{n}\sigma} \sum_{i=1}^{[nt]-[ns]} \varepsilon_i$$

$$= \underbrace{\sqrt{\frac{[nt]-[ns]}{n}}}_{\to \sqrt{t-s}} \cdot \underbrace{\frac{1}{\sqrt{[nt]-[ns]}} \sum_{i=1}^{[nt]-[ns]} \varepsilon_i}_{\overset{d}{\to} N(0,1)}.$$

$$\overset{d}{\to} \sqrt{t-s}\, N(0,1) \overset{d}{=} N(0, t-s).$$

(iii) $\forall\, 0 \le s < t < u \le v \le 1$, $\quad W_n(t) - W_n(s) \perp W_n(v) - W_n(u)$

(iv) $t \to W_n(t)$ is right continuous. $\left(0,\ \frac{\varepsilon_1}{\sqrt{n}\sigma},\ \dots,\ \frac{\varepsilon_1 + \cdots + \varepsilon_{n-1}}{\sqrt{n}\sigma} \right)$

$0 \quad \frac{1}{n} \quad \frac{2}{n} \quad \cdots \quad \frac{n-1}{n} \quad 1$

(v) $\{W_n(s) : 0 \le s \le 1\}$ can be approximated with $\{W(s) : 0 \le s \le 1\}$.

$$\to \begin{cases} W(0)=0. \\ \forall\, 0\le s\le t\le 1, \quad W(t)-W(s) \overset{d}{=} N(0, t-s) \\ \forall\, 0\le s<t\le u<v\le 1, \quad W(t)-W(s) \perp\!\!\!\perp W(v)-W(u). \\ s\to W(s) \text{ is continuous, but nowhere differentiable.} \end{cases}$$

$$T_n = n(\hat\phi - 1)$$

$$= \frac{\sum_{t=1}^{n} W_n\left(\frac{t-1}{n}\right)\left(W_n\left(\frac{t}{n}\right) - W_n\left(\frac{t-1}{n}\right)\right)}{\frac{1}{n}\sum_{t=1}^{n} W_n^2\left(\frac{t-1}{n}\right)}.$$

$$\simeq \frac{\int_0^1 W(s)\, dW(s)}{\int_0^1 W^2(s)\, ds.} = \frac{\frac{1}{2}(W^2(1)-1)}{\int_0^1 W^2(s)\, ds} := T.$$

$|T_n| \overset{d}{\simeq} |T|$ as $n\to\infty$ 이므로, $P(|T_n|>c\,|\,H_0)=\alpha$ 대신

$\simeq P(|T|>c) = \alpha$ 인 c 를 구한다.

(b) Under H_1,

$$T_n = n(\hat\phi - 1) = n(\hat\phi - \phi + (\phi-1))$$

$$= \sqrt{n}\cdot \underbrace{\sqrt{n}(\hat\phi-\phi)}_{\simeq N\left(0, \frac{\sigma^2}{1-\phi^2}\right)} + n(\phi-1)$$

$$\simeq N\left(\sqrt{n}(\phi-1), \frac{\sigma^2}{1-\phi^2}\right).$$

$\Rightarrow P(|T_n| \ge c_\alpha \,|\, H_0) \to 1.$ (T_n is consistent test statistic).

⟨Dickey-Fuller Test⟩.

$$DF_n = \sqrt{\frac{\sum_{t=1}^{n} X_{t-1}^2}{\hat\sigma^2}}\,(\hat\phi - 1). \quad \text{where } \hat\sigma^2 = \frac{1}{n}\sum(X_t - \hat\phi X_{t-1})^2$$

$$\simeq \frac{\sum_{t=1}^{n} W_n\left(\frac{t-1}{n}\right)\left(W_n\left(\frac{t}{n}\right) - W_n\left(\frac{t-1}{n}\right)\right)}{\hat\sigma \sqrt{\frac{1}{n}\sum_{t=1}^{n} W_n^2\left(\frac{t-1}{n}\right)}.}$$

$\sigma \simeq \hat\sigma$

$$\overset{d}{\simeq} DF = \frac{\int_0^1 W^2(s)\, dW(s)}{\sqrt{\int_0^1 W^2(s)\, ds}.} \Rightarrow \frac{1}{2}(W^2(1)-1).$$

reject H_0 if $|DF_n| > c$ ($P(|DF|>c) = \alpha$인 c 찾아서...).

< Augmented D-F Test>: AR(1)이 아닌 AR(p)에서 unit root?

Model : $\phi(B) X_t = \varepsilon_t$. → check $\phi(1) = 0$?

Under H_0 ($\phi(1) = 0$), $\phi(B) = (1-B)\psi(B)$

where $\psi(B) = 1 - \psi_1 B - \cdots - \psi_{p-1} B^{p-1}$

Thus, $\phi(B) X_t = \varepsilon_t$

$\Leftrightarrow (1-B)\psi(B) X_t = \varepsilon_t$

$\Leftrightarrow \psi(B) \Delta X_t = \varepsilon_t$

$\Leftrightarrow \Delta X_t - \psi_1 \Delta X_{t-1} - \cdots - \psi_{p-1} \Delta X_{t-p+1} = \varepsilon_t$.

$\Leftrightarrow X_t = \underset{\uparrow}{} X_{t-1} + \psi_1 \Delta X_{t-1} + \cdots + \psi_{p-1} \Delta X_{t-p-1} + \varepsilon_t$.

$H_0 : \beta = 1, \quad H_1 : \beta \neq 1$. → Under H_0, $\hat\beta \simeq 1$.

($ADF_n = \left| \frac{\hat\beta - 1}{sd(\hat\beta)} \right|$ 고려).

4.3. 모형선택 및 진단.

ARIMA(p,d,q) → p,d,q 결정?

① 모형의 식별 (Box-Jenkins 방법).

(i) 분산 안정적 X ⇒ Box-Cox 변환, log 변환 등...

($f(y) = \frac{y^\lambda - 1}{\lambda}$).

(ii)

	ACF($\rho(k)$)	PACF(ϕ_{kk})
AR(p)	지수적 감소.	p 이후 절단.
MA(q)	q 이후 절단	지수적 감소.
ARMA(p,q)	q-p 이후 소멸	p-q 이후 소멸.

↳ 가설설정) $\begin{cases} H_0 : \phi_{kk} = 0. \\ H_1 : \phi_{kk} \neq 0. \end{cases}$ $\hat\phi_{kk} \sim N(0, \frac{1}{n})$. $\forall k \geq p+1$

⇒ $|\hat\phi_{kk}| > z_{\alpha/2} \frac{1}{\sqrt{n}}$

$\begin{cases} H_0 : \rho(k) = 0 \\ H_1 : \rho(k) \neq 0. \end{cases}$ $\hat\rho(k) \sim N\left(0, \left(1 + 2\sum_{j=1}^{q} \hat\rho^2(j)\right)/n\right)$. $\forall k \geq q+1$.

⇒ $|\hat\rho(k)| > z_{\alpha/2} \sqrt{\cdot}$.

② 모형선택기준.

<u>Def</u> $AIC = -2\log L + k$.

↳ 모수의 개수. $= \dim(H)$.

$\hat{k}$ = argmin AIC(k) 로 선택.

$\hat{k}$ 는 n에 dependent ($\hat{k}_n$). $\left(\begin{array}{l} P(\hat{k}_n \to k_0,\ n\to\infty) \neq 1, \\ P(\hat{k}_n > k_0) > 0. \end{array} \right.$

<u>Def</u> $BIC = -2\log L + \log n \cdot k$. $\left(\begin{array}{l} P(\hat{k}_n \to k_0,\ n\to\infty) = 1, \\ P(\hat{k}_n < k_0) > 0. \end{array} \right.$

③-1 잔차 진단.

$X_t = \phi X_{t-1} + \varepsilon_t \to \hat{\varepsilon}_t \simeq \varepsilon_t$?

<Portmanteau Test>. (Ljung-Box)

- Test of $\{X_t\} \sim ARMA(p,q)$.

$$\left\{ \begin{array}{l} \cdot\ X_t = \phi X_{t-1} + \varepsilon_t \Rightarrow \hat{\varepsilon}_t = X_t - \hat{\phi} X_{t-1}. \\ \cdot\ X_t = \phi_1 X_{t-1} + \cdots + \phi_p X_{t-p} + \varepsilon_t \Rightarrow \hat{\varepsilon}_t = X_t - \hat{\phi}_1 X_{t-1} - \cdots - \hat{\phi}_p X_{t-p} \\ \cdot\ \phi(B) X_t = \theta(B)\varepsilon_t \Rightarrow \varepsilon_t = \sum^{\infty} \pi_j X_{t-j}. \quad \hat{\varepsilon}_t = \sum_{j=0}^{t-1} \hat{\pi}_j X_{t-j}. \end{array} \right.$$

$$\simeq \sum_{j=0}^{t-1} \pi_j X_{t-j},$$

$\Rightarrow$ Obtain $\hat{\rho}_\varepsilon(h) = \hat{\gamma}_\varepsilon(h) / \hat{\gamma}_\varepsilon(0)$. ($\varepsilon_t$들 자기상관)

where $\hat{\gamma}_\varepsilon(h) = \frac{1}{n} \sum_{t=1}^{n-h} (\hat{\varepsilon}_{t+h} - \bar{\varepsilon})(\hat{\varepsilon}_t - \bar{\varepsilon})$. $\quad * \bar{\varepsilon} = \frac{1}{n}\sum \hat{\varepsilon}_t$.

- Define $Q = n(n+2) \sum_{h=1}^{k} (\hat{\rho}_\varepsilon(h))^2 / (n-h)$.

H_0 : $\{X_t\} \sim ARMA(p, q)$

H_1 : $\sim H_0$ $\Rightarrow$ reject H_0 if Q is large.

- Under H_0, $Q \simeq \chi^2_{k-p-q-1}$.

③-2. 과적합진단.

③-3. 정규성검정. $X_t \sim ARMA(p,q) \Rightarrow \phi(B) X_t = \theta(B) X_t$.

$$JB = \frac{n}{6}\left(S^2 + \frac{1}{4}(k-3)^2\right).$$

$H_0 : \varepsilon_t \sim N(0, \sigma^2)$, $H_1 : \sim H_0$. (Under H_0, $JB \simeq \chi^2(2)$).

Ch5. 계절성을 갖는 시계열 모형.

5.1. Seasonal Model.

Assume $X_t = S_t + Z_t$.

(i) $\{S_t\} \perp\!\!\!\perp \{Z_t\}$.

(ii). $S_t = S_{t+s}$ for some $s \ge h$.

$S_1 \dots S_s$ are indep, $ES_t = 0$, $Var(S_t) = \sigma_t^2 < \infty$

(iii) Z_t is stationary. $EZ_t = 0$, $Var(Z_t) = \gamma_Z(0)$

- X_t is not stationary.

$\because$ $Cov(X_{t+h}, X_t) = Cov(S_{t+h} + Z_{t+h}, S_t + Z_t)$

$= E(S_{t+h} S_t) + \gamma_Z(h)$

$$= \begin{cases} \sigma_t^2 + \gamma_Z(h), & h = s, 2s, \dots \\ \gamma_Z(h) & \text{o.w.} \end{cases}$$

- $\nabla_s = 1 - B^s$. $\nabla_s X_t = (1-B^s) X_t = X_t - X_{t-s}$.

$= (S_t + Z_t) - (S_{t+s} - Z_{t+s})$

$= Z_t - Z_{t-s}$. : stationary.

5.2. Seasonal Model w/ Trend.

$$X_t = \underset{\substack{\uparrow \\ \text{추세}}}{T_t} + \underset{\substack{\uparrow \\ \text{계절}}}{S_t} + \underset{\substack{\uparrow \\ \text{정상}}}{Z_t}. \qquad \left(\begin{array}{l} S_t = S_{t+s}, \\ T_t = \beta_0 + \beta_1 t + \dots + \beta_r t^r \end{array} \right)$$

$\Leftrightarrow \nabla_s X_t = \nabla_s T_t + \nabla_s Z_t.$ (S_t 제거).

$\Leftrightarrow (1-B)^{r-1} \nabla_s X_t = \underbrace{(1-B)^{r-1} \nabla_s T_t}_{= \mu} + (1-B)^{r-1} \nabla_s Z_t.$

$\Rightarrow$ ARMA 모형 적합 : $\phi(B) \{(1-B)^{r-1} \nabla_s X_t - \mu\} = \theta(B) \varepsilon_t$.

$\varepsilon_t \sim iid(0, \sigma^2)$.

5.3. 계절 ARIMA 모형 (계절성 가진 ARMA).

Def If $\{X_t\}$ follows the model

$$\Phi(B^s)(1-B^s)^D X_t = \Theta(B^s)\varepsilon_t, \quad \varepsilon_t \sim iid(0,\sigma^2),$$

where $\Phi(z) = 1-\Phi_1 z - \cdots - \Phi_P z^P$, $\Theta(z) = 1+\Theta_1 z + \cdots + \Theta_Q z^Q$. $\left(\begin{array}{l}\Phi(z)\neq 0.\ \forall |z|\le 1,\\ \Theta(z)\neq 0.\ \forall |z|\le 1\end{array}\right)$.

$\{X_t\}$ is said to follow SARIMA(P, D, Q) model.

Ex1. SAR(1) : $X_t = \Phi X_{t-s} + \varepsilon_t$. $|\Phi|<1$.

$$\Leftrightarrow (1-\Phi B^s)X_t = \varepsilon_t.$$

$$X_t = (1-\Phi B^s)^{-1}\varepsilon_t.$$

$$= \sum_{i=0}^{\infty}(\Phi B^s)^i \varepsilon_t.$$

$$= \sum_{i=0}^{\infty}\Phi^i \varepsilon_{t-si}$$

$$\gamma(h) = Cov(X_{t+h}, X_t)$$

$$= Cov\left(\sum_{i=0}^{\infty}\Phi^i\varepsilon_{t-si}\ \sum \Phi^j \varepsilon_{t+h-sj}\right).$$

$$= \sum\sum \Phi^i\Phi^j\, Cov(\varepsilon_{t-si}, \varepsilon_{t+h-sj}).$$

$\hookrightarrow t-si = t+h-sj.$

$j = \underbrace{\frac{h}{s}}_{k} + i.$

$$= \sum_{i=0}^{\infty}\Phi^i\Phi^{i+k}\sigma^2.$$

$$= \begin{cases}\Phi^k \dfrac{\sigma^2}{1-\Phi^2} & h = k\cdot s \quad (s의\ 배수).\\ 0 & o.w.\end{cases}$$

$$\rho(h) = \begin{cases}\Phi^k. & h = k\cdot s.\\ 0 & o.w.\end{cases} = \begin{cases}\Phi^{h/s} & h=0, s, 2s, \ldots\\ 0 & o.w.\end{cases}$$

Ex2 SMA(2) : $X_t = \varepsilon_t + \Theta_1\varepsilon_{t-s} + \Theta_2\varepsilon_{t-2s}$. $|\Theta|<1$.

$$\gamma(h) = Cov(\varepsilon_t + \Theta_1\varepsilon_{t-s} + \Theta_2\varepsilon_{t-2s},\ \varepsilon_{t+h} + \Theta_1\varepsilon_{t+h-s} + \Theta_2\varepsilon_{t+h-2s})$$

$\Rightarrow$. $\left(\begin{array}{l}h=0\\ h=s.\\ h=2s\end{array}\right.$ 나눠서 계산.

5.4. 승법계절모형.

Consider $X_t = S_t + W_t$.

(i) $\{S_t\} \sim SARIMA(P, D, Q)$

(ii) $\{W_t\} \sim ARIMA(\tilde{p}, \tilde{d}, \tilde{q})$. $\rightarrow$ $\tilde{\phi}(B) W_t = \tilde{\theta}(B) \xi_t$ $\rightarrow iid(0, \sigma^2)$

(iii) $S_t \perp W_t$.

- 먼저, $\Phi(B^s) \nabla_s^D S_t = \Theta(B^s) \delta_t$ (제외) $\quad (\delta_t \sim iid(0, \sigma^2))$.

- 따라서, $\Theta^{-1}(B^s) \Phi(B^s) \nabla_s^D X_t$

$= (\quad \text{〃} \quad)(S_t + W_t)$.

$= \delta_t + \Theta^{-1}(B^s) \Phi(B^s) \nabla_s^D W_t$: η_t 라 두자.

$$(1-B)^d \eta_t = (1-B)^d \delta_t + \Theta^{-1}(B^s) \Phi(B^s) \nabla_s^D \underbrace{(1-B)^d W_t}_{= \tilde{\phi}^{-1}(B) \tilde{\theta}(B) \xi_t}$$

$\Rightarrow$ indep. 한 stationary proc. 의 합이므로 stationary.

- By fitting an ARIMA model to $(1-B)^d \eta_t$,

$\phi(B)(1-B)^d \eta_t = \theta(B) \varepsilon_t$.

$\Leftrightarrow \phi(B)(1-B)^d \Theta^{-1}(B^s) \Phi(B^s) \nabla_s^D X_t = \theta(B) \varepsilon_t$.

$\Leftrightarrow \phi(B) \Phi(B^s)(1-B)^d \nabla_s^D X_t = \theta(B) \Theta(B^s) \varepsilon_t$.

Def $\{X_t\} \sim ARIMA(p, d, q)(P, D, Q)_s$ if

Ex1 $X_t = (1 + \theta B)(1 + \Theta B^s) \varepsilon_t$. : $ARIMA(0,0,1)(0,0,1)_s$.

$\Leftrightarrow X_t = \varepsilon_t + \theta \varepsilon_{t-1} + \Theta \varepsilon_{t-s} + \theta\Theta \varepsilon_{t-s-1}$.

- $\gamma(0)$, $\gamma(1)$, $\gamma(2)$, $\gamma(s-1)$, $\gamma(s)$ } 계산.

Ex2 $(1 - \Phi B^s) X_t = (1 + \theta B)(1 + \Theta B^s) \varepsilon_t$. : $ARIMA(0,0,1)(1,0,1)_s$.

$\Leftrightarrow X_t = \Phi X_{t-s} + \varepsilon_t + \theta \varepsilon_{t-1} + \Theta \varepsilon_{t-s} + \theta\Theta \varepsilon_{t-s-1}$.

계산.

$$\begin{cases} Cov(X_t, \varepsilon_t) = \sigma^2, \\ Cov(X_t, \varepsilon_{t-1}) = \theta\sigma^2, \\ Cov(X_t, \varepsilon_{t-2}) = 0 \end{cases} \Rightarrow \begin{cases} \gamma(0). & \gamma(s) \\ \gamma(1) & \gamma(s+1) = \gamma(s-1). \\ \gamma(2) & \gamma(h \geq s+2) = \Phi\gamma(h-s) \end{cases} \cdots$$

Ch6. GARCH 모형. (異분산성 모형).

Def $\{y_t\}$ is called an ARCH process if $\begin{cases} y_t \mid \mathcal{F}_{t-1} \overset{iid}{\sim} (0, \sigma_t^2). \\ \sigma_t^2 = \alpha_0 + \alpha_1 y_{t-1}^2 \quad \alpha_0 > 0, \ 0 \leq \alpha_1 < 1. \end{cases}$

conditional variance.

(or) $\begin{cases} y_t = \sigma_t \xi_t . & \xi_t \sim iid(0,1) \\ \sigma_t^2 = \alpha_0 + \alpha_1 y_{t-1}^2 \end{cases}$

$y_t = f(\xi_t, \xi_{t-1}, \ldots)$

$\Rightarrow y_t \perp\!\!\!\perp \xi_{t+1}, \xi_{t+2}, \ldots$

$\sigma_t^2 = g(\xi_{t-1}, \ldots)$

$\Rightarrow \sigma_t^2 \perp\!\!\!\perp \xi_{t+1}, \ldots, y_t$

- $$\begin{aligned} y_t^2 &= \sigma_t^2 \xi_t^2 \\ &= (\alpha_0 + \alpha_1 y_{t-1}^2)\xi_t^2 \\ &= (\alpha_0 + \alpha_1(\alpha_0 + \alpha_1 y_{t-2}^2)\xi_{t-1}^2)\xi_t^2 \\ &= \cdots = \alpha_0 \sum_{j=0}^{n} \alpha_1^j \xi_t^2 \cdots \xi_{t-j}^2 + \underbrace{\alpha_1^{n+1}}_{\to \sim 0} \xi_t^2 \cdots \xi_{t-n}^2 y_{t-n-1} \\ &\overset{n\to\infty}{=} \alpha_0 \sum_{j=0}^{\infty} \alpha_1^j \xi_t^2 \cdots \xi_{t-j}^2 \quad (\text{converge w.p.} 1) \end{aligned}$$

- $$\begin{aligned} y_t &= \sigma_t \xi_t \\ &= \xi_t \left(\alpha_0 + \alpha_1 \cdot \alpha_0 \sum_{j=0}^{\infty} \alpha_1^j \xi_{t-1}^2 \cdots \xi_{t-j-1}^2\right) \text{로 정의하면} \end{aligned}$$

$\sigma_t = \alpha_0 + \alpha_1 y_{t-1}^2$ 를 만족한다.

- y_t is strictly stationary ($\because$ iid의 함수).
  if $\alpha_0 > 0$, $\alpha_1 \in (0,1)$.

Def $\{y_t\}$ is called ARCH(q), $q \geq 1$ process
if $\begin{cases} y_t = \sigma_t \xi_t & \xi_t \sim iid(0,1) \\ \sigma_t^2 = \alpha_0 + \alpha_1 y_{t-1}^2 + \cdots + \alpha_q y_{t-q}^2 \end{cases}$

$(\alpha_0 > 0, \quad \alpha_1 \ldots \alpha_q \geq 0, \quad \sum \alpha_i < 1).$

- $$\begin{aligned} Var(y_t) &= E\sigma_t^2 \xi_t^2 \\ &= E\sigma_t^2 \\ &= \alpha_0 + \alpha_1 E y_{t-1}^2 + \cdots + \alpha_q E y_{t-q}^2 \end{aligned}$$

$\Leftrightarrow Var(y_t) = \dfrac{\alpha_0}{1 - \alpha_1 - \cdots - \alpha_q}$ ($\because$ stationary).

- $y_t^2 = \alpha_0 + \alpha_1 y_{t-1}^2 + \cdots + \alpha_q y_{t-q}^2 + \nu_t$

$$= \alpha' \underbrace{Y_{t-1}}_{=(1,\, y_{t-1} \dots y_{t-q})^T} + \nu_t .$$

↳ $\hat{\alpha} = \left(\sum_{t=1}^{n} Y_{t-1} Y_{t-1}^T \right)^{-1} \sum_{t=1}^{n} Y_{t-1} y_t^2$ 로..

6.3. GARCH.

Def $\{y_t\}$ is a GARCH process if

$$\begin{cases} y_t = \sigma_t \xi_t , & \xi_t \sim iid(0,1). \\ \sigma_t^2 = \omega + \alpha y_{t-1}^2 + \beta \sigma_{t-1}^2 , & \omega > 0,\ \alpha, \beta \geq 1,\ \alpha + \beta < 1. \end{cases}$$

↳ GARCH(p,q)로 확장: $\sigma_t^2 = \alpha_0 + \alpha_1 y_{t-1}^2 + \cdots + \alpha_p y_{t-p}^2 + \beta_1 \sigma_{t-1}^2 + \cdots + \beta_q \sigma_{t-q}^2$

- ω, α, β estimate? : MLE.

Assume $\xi_t \sim N(0,1)$.

① $f(y_1 \dots y_n \mid y_0, \sigma_0) = f(y_n \mid y_{n-1}, y_{n-2}, \dots y_1, y_0, \sigma_0)$
$\times f(y_{n-1} \mid y_{n-2} \dots y_1 \ \text{"}\ )$
$\vdots$
$\times f(y_1 \mid y_0, \sigma_0)$

$$= \frac{1}{\sqrt{2\pi}\sigma_n} e^{-\frac{y_n^2}{2\sigma_n^2}} \times \cdots \times \frac{1}{\sqrt{2\pi}\sigma_1} e^{-\frac{y_1^2}{2\sigma_1^2}}$$

$\left(\begin{array}{l} \hat{\sigma}_t^2 = \omega + \alpha y_{t-1}^2 + \beta \sigma_{t-1}^2 \\ : \sigma_0, y_0 \text{ 으로부터 recursive..} \end{array} \right)$

② $L(\theta) = \text{Log } f(\quad)$

$$= -\sum_{t=1}^{n} \left(\log \hat{\sigma}_{t(\theta)}^2 + \frac{y_t^2}{\hat{\sigma}_t^2(\theta)} \right) + C.$$

$\theta = (\omega, \alpha, \beta)$

↳ maximize 하는 θ.

<Threshold GARCH>.

$$\begin{cases} y_t = \sigma_t \xi_t . \\ \sigma_t^2 = \omega + \alpha_{11} (y_{t-1}^{+})^2 + \alpha_{12} (y_{t-1}^{-})^2 + \beta_1 \sigma_{t-1}^2 \ \dots \end{cases}$$

6.4. 정규성 검정. (J-B)

$$\hat{T}_{JB} = n \left(\frac{S_n^2}{6} + \frac{(K_n - 3)^2}{24} \right).$$

$H_0 : \xi_t \sim N(0,1)$

$H_1 : \sim H_0 .$

Under H_0, $\hat{T}_{JB} \sim \chi^2(2)$. as $n \to \infty$

+) Ljung-Box Test.

6.6. Value-at-Risk.

< Quantile Estimation >.

<u>Def</u> If $X_1 \dots X_n \overset{iid}{\sim} F$, $q_X(\tau) := F^{-1}(\tau)$. $0<\tau<1$.

$= \inf\{x : F(x) \ge \tau\}$ → quantile of X at τ.

$= \inf\{x : P(X \le x) \ge \tau\}$.

$$\Rightarrow q_{aX+b}(\tau) = F_{aX+b}^{-1}(\tau)$$
$$= \inf\{y : P(aX+b \le y) \ge \tau\}.$$
$$= \inf\{ax+b : P(aX+b \le ax+b) \ge \tau\}.$$
$$= a \inf\{x : P(X \le x) \ge \tau\} + b.$$
$$= a \cdot q_X(\tau) + b.$$

<u>다른 Def</u> $F^{-1}(\tau) = \min\{m : E\rho_\tau(X-m)\}$. $\rho_\tau(x) = x(\tau - I_{(x>0)})$

$\hat{\rho}_\tau(X-m) = \frac{1}{n}\sum \rho_\tau(X_t - m)$ 으로..

— Consider GARCH model

: $\begin{cases} y_t = \sigma_t \varepsilon_t, & \varepsilon_t \sim iid(0, w), \quad w>0. \\ \sigma_t^2 = 1 + \alpha y_{t-1}^2 + \beta \sigma_{t-1}^2, & \alpha, \beta \ge 0, \quad \alpha+\beta<1. \end{cases}$

→ Given I_{t-1},

$$q_{y_t|I_{t-1}}(\tau) = \operatorname{argmin} E(\underbrace{\rho_\tau(y_t - m) | I_{t-1}}_{(*)}).$$

$$(*) = \sigma_t \, q_\varepsilon(\tau).$$

known given I_{t-1}. ($\sigma_t^2 = 1 + \alpha y_{t-1}^2 + \beta\sigma_{t-1}^2$) Let ξ_t.

$$= \sqrt{1 + \alpha y_{t-1}^2 + \beta \sigma_{t-1}^2} \cdot \xi.$$

$$= (\alpha, \beta, \xi)\text{의 함수}.$$

$$\operatorname{arg\,min} \frac{1}{n}\sum_{t=1}^n \rho_\tau(y_t - \underbrace{\hat{q}_t(\omega, \alpha, \beta)}_{=\hat{\sigma}_t \cdot \xi}).$$

6.5. 변화점 탐지.

① Mean change Test.

$X_1 \dots X_n \overset{indep}{\sim} N(\mu_i, \sigma_0^2)$. $i = 1, \dots, n$.

$\begin{cases} H_0 : \mu_1 = \dots = \mu_n = \mu_0. \\ H_1 : \mu_1 = \dots = \mu_k \ne \mu_{k+1} = \dots = \mu_n. \end{cases}$

- Define $T_n(k) = \frac{1}{\sqrt{n}\,\sigma_0} \left| \sum_{i=1}^{k} X_i - \left(\frac{k}{n}\right) \sum_{i=1}^{k} X_i \right|$. $k = 1, \dots, n$.

$\rightarrow T_n = \max_{1 \le k \le n} T_n(k)$.

- Let $W_n(s) = \frac{1}{\sqrt{n}\,\sigma_0} \sum_{i=1}^{[ns]} (X_i - \mu_0)$. $0 \le s \le 1$.

$\{W_n(s)\} \simeq \{W(s)\}$ under H_0!

$T_n([ns]) = \left| W_n(s) - \underbrace{\frac{[ns]}{n}}_{\simeq s} W_n(1) \right| \overset{d}{\approx} |W(s) - sW(1)|$ 이므로

$:= W^0(s)$. (brownian bridge)

$\sup T_n([ns]) = \sup |W(s) - sW(1)|$,

$\max T_n(k) = T_n \overset{d}{\simeq} \sup |W^0(s)|$.

$\hookrightarrow$ reject H_0 if $T_n > C_\alpha$.

C_α : $P(\sup_{0 \le s \le 1} |W^0(s)| > C_\alpha) = \alpha$

이때 $\underset{1 \le k \le n}{\operatorname{argmax}} T_n(k) = \hat{k}_n$.

$\hookrightarrow$ 변화점.

② Variance change

$X_i \sim N(\mu_0, \sigma_i^2)$, $i = 1, \dots, n$

$$\begin{cases} H_0 : \sigma_1^2 = \dots = \sigma_n^2 = \sigma_0^2 \\ H_1 : \sigma_1^2 = \dots = \sigma_k^2 = \sigma_0^2 \ne \sigma_0'^2 = \sigma_{k+1}^2 = \dots = \sigma_n^2 \end{cases}$$

$$T_n = \max_{1 \le k \le n} \frac{1}{\sqrt{n}\,\hat{\tau}_n} \left| \sum_{i=1}^{k} X_i^2 - \frac{k}{n} \sum_{i=1}^{k} X_i^2 \right|$$

$$\overset{d}{\simeq} \sup |W^0(s)|.$$

Ex. GARCH (1,1). $\hat{\xi}_t = y_t / \hat{\sigma}_t$

$\hookrightarrow$ X_i 에 $\hat{\xi}_t$ 대입. . . .

Note : $E y_t^2 = \frac{w}{1 - \alpha - \beta}$.

Ch7. 다변량 시계열.

Def $\{\underline{Y_t} = (Y_{1t}, \dots, Y_{mt})^T, t \in \mathbb{Z}\}$ is stationary

if (i) $\forall t$, $E\underline{Y_t} = \underline{\mu}$

(ii) $\forall t$, $E\|Y_t\|^2 < \infty$

(iii) $\forall i,j,h$, $Cov(Y_{i,t}, Y_{j,t+h}) = Cov(Y_{i,0}, Y_{j,h})$.

Def $\{Y_t\}$ is strictly stationary if

$(\underline{Y_1}^T, \dots, \underline{Y_t}^T) \overset{d}{=} (\underline{Y_{1+h}}, \dots, \underline{Y_{t+h}})$

Def ACF : $\Gamma(h) = E(\underline{Y_t} - \underline{\mu})(\underline{Y_{t+h}} - \underline{\mu})^T$

$= (\Gamma_{ij}(h))$

where $\Gamma_{i,j}(h) = Cov(Y_{i,t}, Y_{j,t+h})$.

- (i) $\Gamma(-h) = \Gamma(h)$

(ii) $|\Gamma_{ij}(h)| \le \sqrt{\Gamma_{ii}(0)}\sqrt{\Gamma_{jj}(0)}$ → Y_t의 차원.

(iii) $\sum_{i,j=1}^{k} \underline{a_i}^T \Gamma(i-j)\underline{a_j} \ge 0$. $\forall a_i \in R^{m}$

Def $\{Y_t\}$ follows VARMA(p, q) if

$\underline{Y_t} - \underline{\Phi_1}\,\underline{Y_{t-1}} - \dots - \underline{\Phi_p}\,\underline{Y_{t-p}} = \underline{\varepsilon_t} + \underline{\Theta_1}\,\underline{\varepsilon_{t-1}} + \dots + \underline{\Theta_q}\,\underline{\varepsilon_{t-q}}$

$\{\varepsilon_t\} \sim iid\ (0, \Sigma)$.

$\underline{\Phi_1}, \dots \underline{\Phi_p}, \underline{\Theta_1}, \dots \underline{\Theta_q}$: $m \times m$ matrix

- If $\det(\underline{I} - \underline{\Phi_1} z - \dots \underline{\Phi_p} z^p) \ne 0$ $\forall |z| \le 1$,

Y_t is stationary.

$\Rightarrow \Phi(B) Y_t = \Theta(B)\varepsilon_t$.

$Y_t = \Phi^{-1}(B)\Theta(B)\varepsilon_t = \sum_{j=0}^{\infty} C_j \varepsilon_{t-j}$.

$\Leftrightarrow \Theta(B) = \Phi(B)\cdot C(B)$ (항등식 비교)

where $C(B) = \sum_{j=0}^{\infty} C_j B^j$.

$- \Gamma(h) = E\, Y_t Y_{t+h}'$

$$= E \sum_{i=0}^{\infty} C_i \varepsilon_{t-i} \cdot \left(\sum_{j=0}^{\infty} C_j \varepsilon_{t+h-j}\right)^T$$

$$= \sum\sum C_i \, E\, \varepsilon_{t-i} \varepsilon_{t+h-j} C_j^T \quad j = i+h$$

$$= \sum \left(C_i \sum C_{i+h}^T\right)$$

$\rightarrow \hat{\Gamma}(h) = \frac{1}{n} \sum_{t=1}^{n-h} (Y_t - \bar{Y})(Y_{t+h} - \bar{Y})^T$

$- \; n \| \bar{Y} - \mu \| \longrightarrow \sum_{i=1}^{m} \sum_{h=-\infty}^{\infty} \gamma_{ii}(h).$

<u>Def</u> If $\{Y_t\}$ is non-stationary,

but $\beta^T Y_t$ is stationary, $\{Y_t\}$ is said to be co-integrated.

<u>Ex</u> $\begin{cases} Y_{1t} = Y_{1,t-1} + \varepsilon_{1t} \\ Y_{2t} = Y_{1,t-1} + \varepsilon_{2t} \end{cases}$ $\quad \underset{\sim}{\varepsilon_t} = \begin{pmatrix} \varepsilon_{1t} \\ \varepsilon_{2t} \end{pmatrix}.$ iid.

$\rightarrow (1 \; -1)\begin{pmatrix} Y_{1t} \\ Y_{2t} \end{pmatrix} = \varepsilon_{1t} - \varepsilon_{2t} \Rightarrow$ Stationary.

↳ VECM으로 분석.

11.2. 추론

$Var(p)$: $Y_t = \Phi_1 Y_{t-1} + \cdots + \Phi_p Y_{t-p} + \varepsilon_t$

$\det(I - \Phi_1 z - \cdots - \Phi_p z^p) \neq 0, \quad \forall |z| \leq 1.$

$\rightarrow \sum_{i=1}^{n} \| Y_t - \sum_{i=1}^{p} \Phi_i Y_{t-i} \|$ 최소화하는 … $\Rightarrow \begin{cases} \hat{\Sigma} = \frac{1}{n} \sum \hat{\varepsilon}_t \hat{\varepsilon}_t' \\ \hat{\varepsilon}_t = Y_t - \hat{\Phi}_1 Y_{t-1} - \cdots - \hat{\Phi}_p Y_{t-p}. \end{cases}$

$-$ AIC : $\log|\hat{\Sigma}| + \frac{2}{n} m^2 p.$

BIC : $\log|\hat{\Sigma}| + \frac{\log n}{n} m^2 p.$

$- \quad Q = n^2 \sum_{h=1}^{H} \frac{1}{n-h} \mathrm{tr}\left(\hat{\Gamma}(h) \hat{\Gamma}^{-1}(0) \hat{\Gamma}(h) \hat{\Gamma}^{-1}(0)\right)$

↳ $\frac{1}{n} \sum_{t=1}^{n-h} \hat{\varepsilon}_{t+h} \hat{\varepsilon}_t^T$

$\overset{d}{\sim} \chi^2(m^2(H-p)).$